Physics of Noise

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Noise sources in measuring instruments

When we record a signal, several noise sources will come to perturb the signal .

- The intrinsic system noise. For a well-designed acquisition system, this is this contribution that must dominate the measure.
- The noise related to the amplification. All amplifiers make some noise in their output signal, even if no signal is present at their input. One must adapt the amplifier so that its own noise is minimal compared with the signal.
- Digitizing noise. This is the noise introduced upon converting the analog signal to digital form and in the spectrum folding or aliasing. To minimize this noise, you must use superior Analog to Digital converters and provide them with efficient anti-aliasing filters.

Intrinsic noise sources of a physical system

Most physical systems that we use, measure a value related to the large number of physical particles.

- Noise related to a thermodynamics property which is associated to particles motions. These noises have an intensity proportional to the absolute temperature T. (For instance, the noise of a resistor).
- The noise associated with the fine grain nature of matter. If we consider a particle flux, as long as their motions are uncorrelated, lead to the shot noise. This noise is temperature independent. (For instance, the rain noise).
- Noise having a $1/f$ spectrum. A large number of natural phenomena present fluctuations with a spectrum like $1/f^{\alpha}$ (The height of water versus time for a river for instance).

Thermodynamics noises

All the thermodynamics variables present thermal fluctuations:

- Voltage fluctuations at the pole of a resistor.
- Brownian motion.
- Pressure fluctuations.
- etc.

Prediction of these fluctuations relies on two relations:

- The Einstein relation
- The fluctuations dissipation theorem

The Einstein relation

Tells us that each degree of freedom has a mean thermal energy equals to: $\frac{1}{2}k_B.T$ avec $k_B = \frac{R}{N_a} = 1.3806210^{-23} J/K$.
If we measure the position x of a particle maintained by a spring of stiffness k. The position x Will display fluctuations δx such that:

$$\frac{1}{2}k. < \delta x^2 >= \frac{1}{2}k_B.T$$

.

If we consider the dipole made of a resistor R in parallel with a condenser C, the RC circuit will display voltage fluctuations δV such that:

$$\frac{1}{2}C. < \delta V^2 >= \frac{1}{2}k_B.T$$

.

- Be careful that the Einstein relation applies in real space and assumes that the fluctuating variables do not suffer any filtering (which is difficult to achieve in experiments).

The fluctuations dissipation theorem

It indicates that we may describe the fluctuations of a system by a random perturbation f_n apply to this system. In Fourier's space f_n is a white noise that will be filtered by the spectral response of the system. This theorem links the thermodynamics fluctuations of the system to the dissipative elements that it contains. That is to say, that f_n only depends of the dissipative terms of the system.

If we consider two examples: the RC circuit and the system of a micro-beads in a viscous media (water) tethered by a spring. The fluctuations dissipation theorem tells us that the element responsible of the fluctuations are those which dissipates the energy. In practice, this is here R for the RC circuit and the viscous drag one $6\pi\eta.r$ of the micro-bead.

This sounds to contradict the Einstein relation which pointed out that fluctuations were related to C and k not the other terms.

Fluctuations of the micro-bead tethered by a spring

Equation of motion:

$$k.x(t) + 6\pi\eta.r\dot{x}(t) = F_L(t)$$

Where the autocorrelation function of $F_L(t)$ is such that $< F_l(t).F_l(0) >= \delta(t)$ that is $\tilde{F}_l(\omega) = cte$. We are trying to compute $F_L(t)$ while using the Einstein relation:

$$\frac{1}{2}k. < \delta x^2 >= \frac{1}{2}k_B.T$$

FT of the equation of motion: $k.\tilde{x} + i\omega 6\pi\eta.r\tilde{x} = \tilde{F_L}$ That is $\tilde{x} = \frac{\tilde{F_L}}{k+i\omega 6\pi\eta.r}$ ou encore $\tilde{x}^2 = \frac{\tilde{F_L}^2/k^2}{1+\omega^2/\omega_c^2}$ avec $\omega_c = \frac{k}{6\pi\eta.r}$

Fluctuations of the micro-bead tethered by a spring...

The Parceval theorem allows us to compute $x^2(t)$:

$$\int_{-\infty}^{\infty} x^2(t)dt = \int_0^{\infty} \tilde{x}^2(\omega)d\omega \ (\omega >= 0 \text{ since } x(t) \text{ real})$$

$\int_0^{\infty} \frac{F_L^2}{k^2} \frac{1}{1+(\omega/\omega_c)^2} d\omega = \frac{F_L^2}{k^2} \int_0^{\infty} \frac{1}{1+y^2} \omega_c dy$. With $y = \frac{\omega}{\omega_c}$ and $dy = \frac{d\omega}{\omega_c}$. $\int_{-\infty}^{\infty} x^2(t)dt = \omega_c \frac{F_L^2}{k^2}[arctan(y)]_0^{\infty} = \frac{\pi}{2}\frac{F_L^2}{k.6\pi\eta r} = \frac{k_B T}{k}$

$$\tilde{F}_L^2(\omega) = \frac{2}{\pi}(6\pi\eta r).k_B T \text{ ou } \tilde{F}_L^2(f) = 4(6\pi\eta r).k_B T$$

For the RC circuit one obtains:

$$\tilde{E}_n^2(\omega) = \frac{2}{\pi}.R.k_B T \text{ where } \tilde{E}_n^2(f) = 4.R.k_B T$$

The left equation is the noise density per pulsation, the right one is the noise density per Hertz unit.

Measuring force using Brownian motion

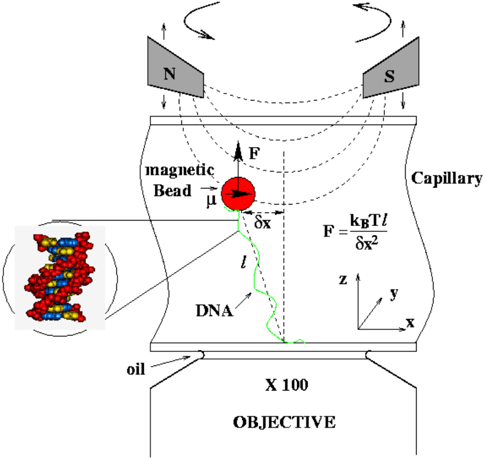


Figure : *Principle of Force measurement.* $< x^2 >$ *Allows us to estimate the stiffness* k*. Since* $k = F/l$*, it is also possible to measure the viscous friction by measuring* f_c *or* τ_c*. This method of force calibration relies solely on measuring distances.*

Can be applied to AFM Calibration or optical tweezers.

Position fluctuations as a function of the stiffness.

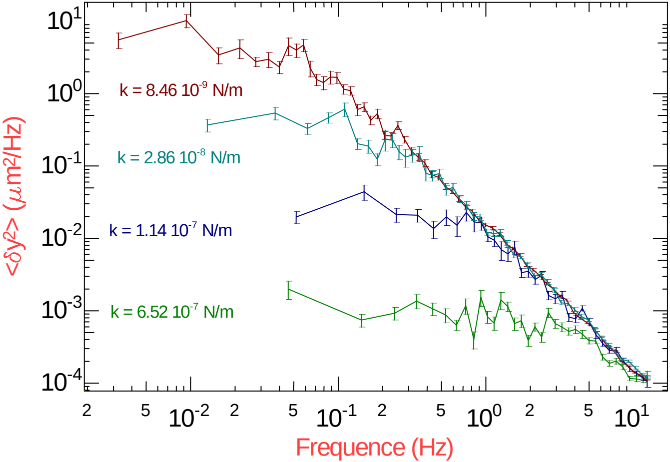


Figure : *Brownian fluctuations of a bead versus the stiffness k of the spring. At low frequencies, the fluctuations amplitude is proportional to k. At high frequencies, the viscous drag limit the fluctuations.*

Strength of the Langevin force versus the stiffness of the system.

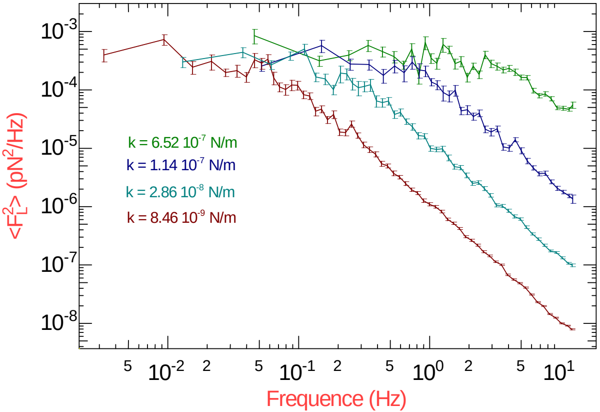


Figure : *Langevin Force measured via the position fluctuations multiplied by the stiffness of the spring k. Since the Langevin force does depend solely on viscous terms, it is independent of the k. On the other hand, the band width of the system does depend on k.*

RMS noise and signals bad width.

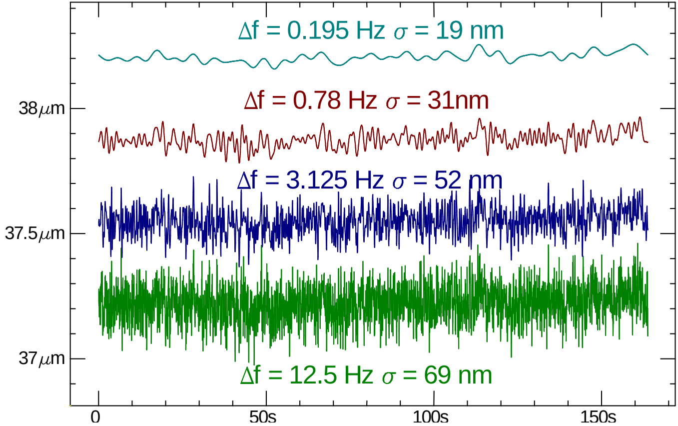


Figure : *Relation between the bandwidth Δf of a signal (here Brownian motion $f_c = 4Hz$) and its RMS excursions. We have $\sigma \propto \sqrt{\Delta f}$. Here the same signal has been represented with variable filtering (offset were added for clarity).*

RMS noise and signal bad width...

The Langevin force F_L or the noise voltage on a resistor E_n is in fact noise densities in Fourier's space. Their physical units are respectively $\mathrm{pN}/\sqrt{\mathrm{Hz}}$ and $\mathrm{n}V/\sqrt{\mathrm{Hz}}$. In order to compare experimental noise, one needs to compare their spectral noise density. The RMS (Root Mean Square) value in direct space may be confusing, since bandwidth is not often the same which is not easy to realize upon seeing the signal. One way to compare signals in real space consists of filtering them first so that they have the same bandwidth.

Noise from a resistor R.

A resistor $R \Rightarrow$ is a generator of noise $\tilde{E}_n^2(f) = 4.R.k_B T$. If we consider signals in real space $E_n^2(t) = 4.R.k_B T \Delta f$. Most sensors rely on electronics and they start by amplifying the sensor signal. These sensors may often be modeled by a resistor R (in some case an impedance). The value of this resistor and its temperature define the intrinsic noise of the system. Order of magnitude: $R = 10k\Omega$ and $T = 300^\circ K$ leads to Ãă 12 $nV/\sqrt{Hz}$, for a 50Ω which is the characteristic impedance of coaxial cables this leads to 0.82 $nV/\sqrt{Hz}$. Consider a scope $R = 1M\Omega$, $C = 10pF \Rightarrow \Delta f = 16kHz$ Corresponds to $\approx 15\mu V$ of RMS noise $R = 50\Omega$, $\Delta f = 1GHz$ corresponds to $\approx 26\mu V$. Maximal frequency: $\hbar\omega = k_B T$, corresponds to infrared room temperature. Working at low temperature, one can observe frequency cutoffs $25mK \Rightarrow 1GHz$.

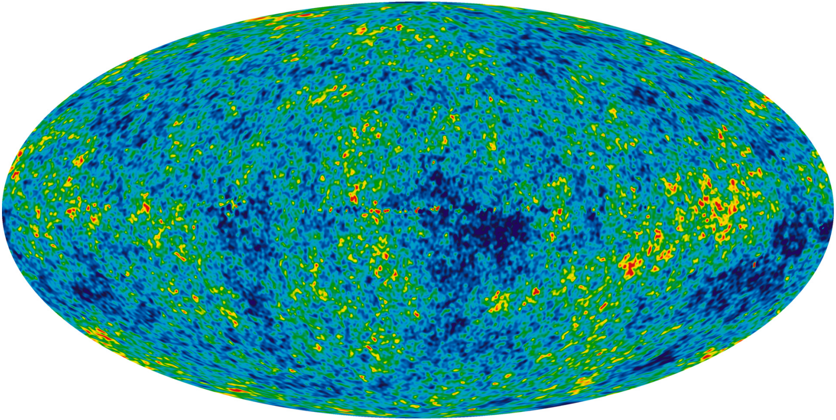
Noise of a resistor R, a temperature measurement

If one measures the noise of a resistor with a very good amplifier, it is possible to deduce the absolute temperature of the system. This is not a very sensitive thermometer but it is an absolute one.

Figure : *In 1965, Penzias and Wilson at Bell Labs. Have use this technic to measure the noise of a microwave antenna pointed in the sky. Penzias and Wilson have designed a sensor with a very low intrinsic noise.*

To their surprise, they observe an extra noise that they first cannot explain. Finally, they will discover that this noise is a leftover of the Big bang, some electromagnetic noise fluctuations having a $3^{\circ}K$ temperature. (Meaning that their sensor had a noise temperature well below this value). Penzias and Wilson were able to provide an experimental validation of the Big-bang theory and they received the Nobel Prize. .

Since that the Planck satellite, measures the electromagnetic fluctuations anisotropy with extremely high accuracy to understand the universe formation



Shot noises

If one observes cars passing on the highway without congestion, the time separating two consecutive cars nearly follows a Poisson distribution. This distribution is very noisy, if you want to estimate the car flow from cars counting, one will notice that your estimation will present a strong noise proportional to the square root of the car number that you counted. This noise is in fact the Shot noise, it is characterized by. $\sigma_I^2 \propto I$ You shall observe this noise in many systems:

- A flow of electrons in vacuum
- In a flow of photons occurring in light measurement
- whenever electrons cross a potential energy barrier (like in a diode)
- etc.

This noise was discovered by Schotky and often his name is used to qualify this noise

Shot noise in electrical currents

As soon as in a system the electrons flux is controlled through the crossing of a potential barrier, one observes shot noise. If it occurs for an electron current, it takes the form:

$$i_n^2(t) = 2I.e.\Delta f \text{ where } i_n^2(f) = 2I.e$$

This is a white-frequency noise that is independent of the temperature. Its strength is proportional to the charge of the carrier e. For a $1\mu A$ the noise corresponds to $0.6pA/\sqrt{Hz}$. If one considers a photomultiplier or an avalanche photodiode, each time a photon or an electron triggers the system, N electrons are generated. The shot noise is thus reinforced:

$$i_n^2(f) = 2(n.q).q = 2nq^2 \text{ oÃź } q = N.e$$

Other examples of shot noise

Most light sensors are shot noise limited. Some light sensors allow single photon detection.

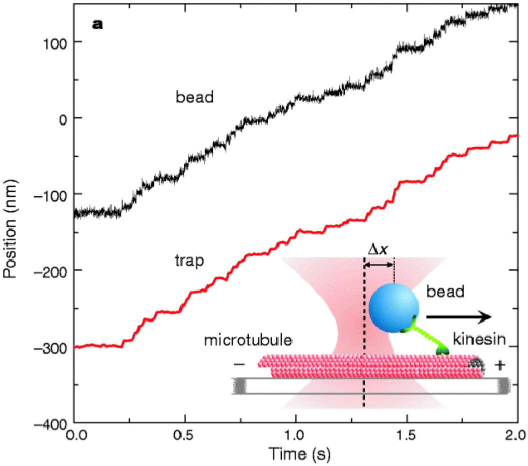


Figure : *Displacement of a molecular motor: the kinesin along a microtubule. This motor advanced by steps of 8 nm having a Poisson step distribution in time. The position fluctuations correspond to the integral of a shot noise.*

Number of photons impacting a pixel in a camera

0.001, 0.01, 0.1; 1,10,100,1000,10000,100000 ph/pixel "Photon noise" by Mdf - Photon-noise.jpg. Licensed under CC BY-SA 3.0 via Wikimedia Commons -

The existence of events correlation reduces shot noise

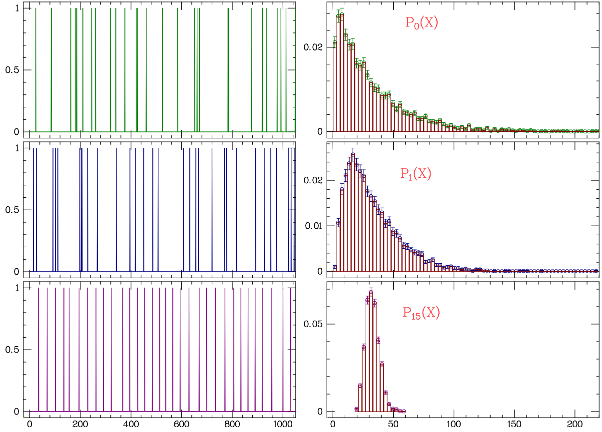


Figure : *On the left, simulation of an event flux with a complete random nature (Poisson distribution (green)) towards a situation nearly periodic (magenta). On the right, histograms of the dwell time between events.*

The existence of events correlation reduces shot noise

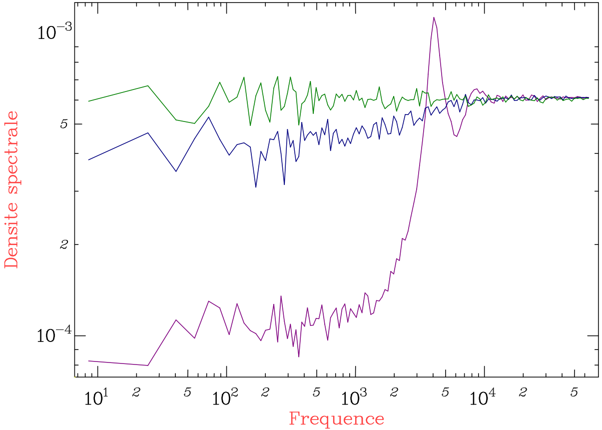


Figure : *Simulations of power spectrum of the previous signals. When the flux corresponds to the pure random case (Poisson distribution in green), the noise density is white. When the situation is nearly periodic (magenta) the shot noise is depressed at low frequencies.*

Calibrating camera sensitivity in electrons using shot noise

If you want to determine how many electrons produce a specific signal on a camera, one can study its shot noise. For instance, a simple CCD camera (JAI-AV10), provides a digitized video signal varying between 0 and 255 (8 bits) 60 times per second proportional to the light level. If we illuminate this camera with an LED which power may be varied from 0 to 100 in arbitrary units. For a given homogeneous illumination, we measure the signal $< S >$ per pixel and also its variance $< S - < S >>^2$. These results are provided in the figure 10. If the camera was perfect, the signal S would be proportional to the number of photons n which are converted in electrons for each pixel: $S = \alpha . n$. We propose to determine α.

Calibrating camera sensitivity in electrons using shot noise

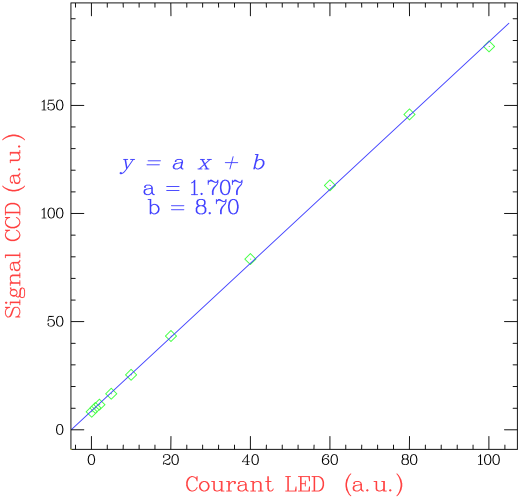


Figure : *Average signal measured per pixel versus the illumination intensity. We observe that the output signal is linear with the illumination intensity. As the camera can only display positive signals, the CCD signal is slightly shifted to a positive value (here 8.7) when the camera is in complete dark.*

Calibrating camera sensitivity in electrons using shot noise

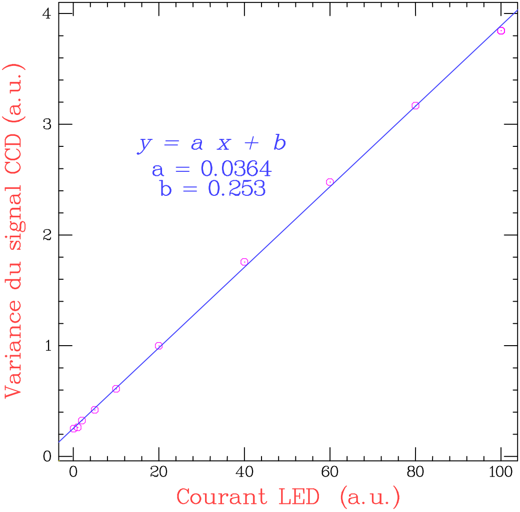


Figure : *Signal variance of each pixel for the camera as a function of the LED current. The slope of this line characterizes the shot noise of the camera. The origin ordinate is the intrinsic noise of the camera, that is to say the noise of the camera in the dark.*

Calibrating camera sensitivity in electrons using shot noise

- Express the signal variance versus α and n. *var(S)=* $\alpha^2 n$ *where the standard error* $\sigma = \alpha\sqrt{n}$.
- Deduce the value of α. How many electrons are detected in each pixel when the illumination is set to 100 (the camera is then nearly saturated). *We have* $S(I) = a_S I = \alpha n$ *and* $var(S(I)) = a_v I = \alpha^2 n$ *thus we deduce that* $\alpha = a_v/a_S = 0.0364/1.707 = 0.0213$. *Moreover, we have* $n = (a_S/\alpha).I$ *thus* $n = 80.05 \times I$ *for* $I = 100$ *each pixel accumulate 8005 photons (we assume that each photon hitting a pixel is converted to an electron with a quantum efficiency of 1)*.
- To what photon number does the camera noise corresponds to (per image). *As the camera noise at* $I = 0$ *corresponds to a variance of 0.253, we have* $n_0 = var(S(I = 0))/\alpha^2$, *thus it is equivalent to 560 photons!*

There is no shot noise in a resistance bridge

Be careful to always add noise power (and not amplitudes)!

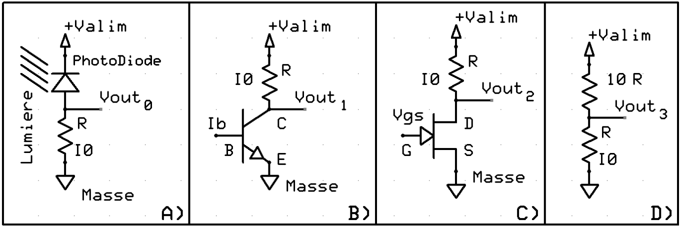


Figure : *Schematic of four circuits in which the same current I_0 flows through a resistance R but which output voltages do not reflect the same noise. In the first three (A,B,C) the current is driven by a semiconductor element which let the charges flow one by one. The current I_0 displays shot noise. In the case D), the $10R$ resistor is far bigger than R and combined with the power supply V_{alim} acts as a current source limiting it to I_0. But this circuit does not display shot noise but only the resistor noise.* $V^2_{out1,2,3} = (4k_B T.R. + 2I_0.e.R^2)\Delta f$ *and* $V^2_{out4} = (4k_B T.R.)\Delta f$

An amplifier presents two noise sources: one in voltage, one in current.

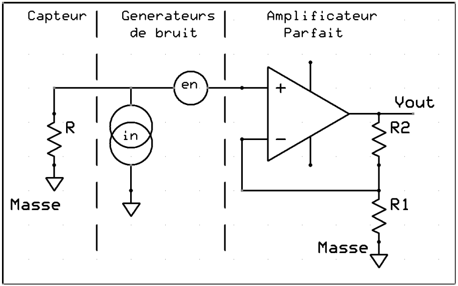


Figure : *Principle schematic of an amplifier chain. The sensor is modeled by a resistance R connected to the amplifier. This amplifier is modeled by a perfect amplifier associated to two noise sources: a voltage one e_n and a current one i_n.*

Setting $R = 0$, the amplifier presents the noise e_n^2. If R is very large, the amplifier noise is $\propto R^2 i_n^2$.

$$V_{out}^2 = G^2(e_n^2 + R^2 i_n^2 + 4k_B T.R).\Delta f$$

Minimization the amplification noise depends on the sensor resistance

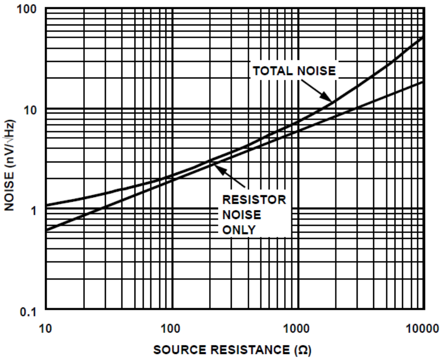


Figure : *Noise of a resistive sensor R connected to a AD797 amplifier. If R is very small, the voltage noise source e_n dominates, if R is very large this is the current contribution which now dominates noise $R.i_n$. For given e_n and i_n, there is an optimum $R_o = e_n/i_n$.*

One characterizes an amplifier by its noise figure. For the AD797, $e_n = 1nV/\sqrt{Hz}$ and $i_n = 5pA/\sqrt{Hz}$

$$F = \frac{SNR_{out}}{SNR_{in}} = \frac{(e_n^2 + R^2 i_n^2 + 4k_B T.R)}{(4k_B T.R)} = 1.6 \text{ (2dB) for } R_0 = 200\Omega$$

Noise temperature of an amplifier

It is possible to define a noise temperature of an amplifier, this would be the noise equivalent on its optimum resistor R_o connected to its input assuming that this resistance is at zero temperature. In these conditions, the noise is described on the resistor R_o as if it was at a temperature T_A. A good amplifier has a very low noise temperature. For the AD797, $e_n = 1nV/\sqrt{Hz}$ and $i_n = 5pA/\sqrt{Hz}$

$$T_A = \frac{(e_n^2 + R^2 i_n^2)}{(4k_B.R)} = 183K \text{ for } R_o = 200\Omega$$

For the AD 745, $e_n = 3.2nV/\sqrt{Hz}$ and $i_n = 6.9fA/\sqrt{Hz}$, $R_o = 464k\Omega$

$$T_A = \frac{(e_n^2 + R^2 i_n^2)}{(4k_B.R)} = 0.807K \text{ for } R_o = 464k\Omega$$

The AD797 has its first amplification stage build using bipolar NPN transistors, the AD745 uses JFET transistors.

Physics behind the noise generator in an amplifier

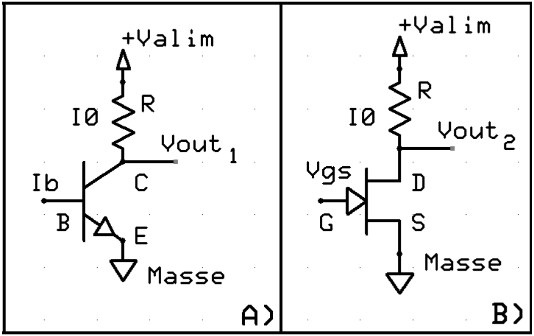


Figure : *The first stage of an amplifier is made of a differential transistor pair that is made either of bipolar or JFET or MOS transistors. This element regulates the current flow I_0 by the voltage applied either to their bases or their gates.*

The voltage noise e_n is produced mainly by the shot noise occurring in I_0. These transistors have small polarization currents I_b or I_g which shot noise is the main contribution in i_n. Junction transistors: display strong transconductance leading to small e_n. But has significant I_b (μA) and thus have strong i_n. For JFETs and MOS this is the just opposite: moderate e_n but very weak I_g (pA) leading to very small i_n.

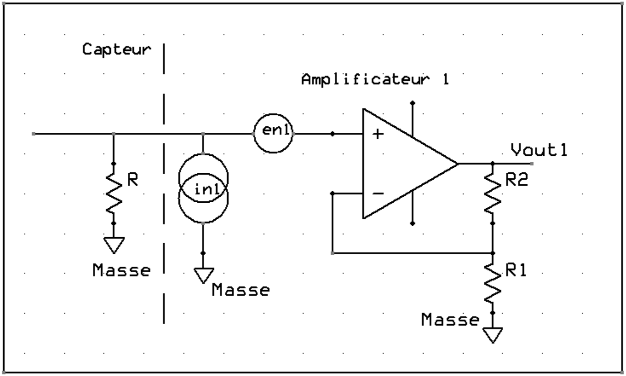


Figure : *Principle schematic of a resistor noise measurement using an amplifier. Where e_n is the voltage noise generator and i_n is the current noise generator of the amplifier.*

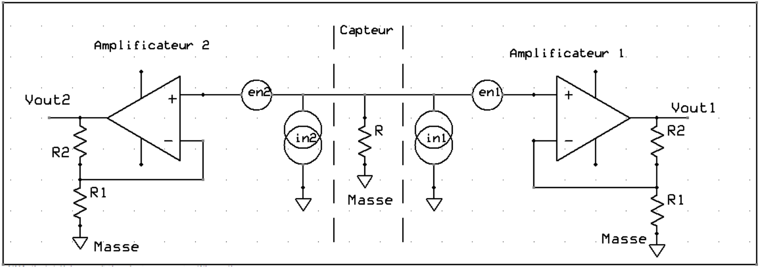


Figure : *Principle schematic of a resistor noise measurement using two amplifiers. Where e_{n1} and e_{n2} are the voltage generators and i_{n1} and i_{n2} are the current generators of the amplifiers.*

Frequency dependence of the noise of an amplifier

We observe a 1/f component in nearly all systems. For a resistor, this is a noise where the resistor value presents slow frequency modulations. The exact understanding is still missing but impurities seem to play a role by modulating the resistor value between two values associated at different states.

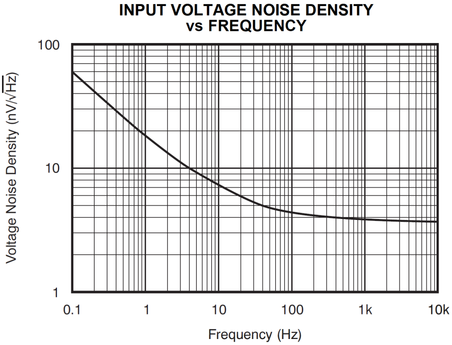


Figure : *Evolution of the noise generator e_n of a JFET amplifier with frequency. The noise level is constant at high frequencies but continuously increases as frequency goes down. The Current noise contribution also presents a 1/f component, but with a different position in frequency.*

M.B. Weissman, Rev. Mod. Phs. (1988) **60**-2, p. 537-71

Noise can reveal quantum coherence

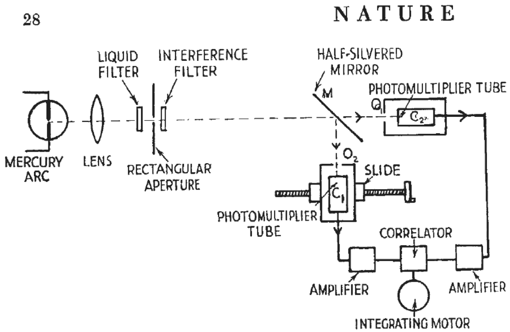


Fig. 2. Simplified diagram of the apparatus

Figure : *In the de Hanbury-Brown Twiss experiment: a light beam is divided in two before arriving on two detectors which signals are correlated. If the light comes a thermal source, a positive correlation is observed (Photons bunching). If the light source sends it photons one by one, we observe an anti-correlation.*

Hanbury Brown, R. and Twiss, R. Q. Correlation between photons in two coherent beams of light. Nature 177, 27-29 (1956)