Fourier transform

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Signal decomposition on a base of orthogonal polynomials.

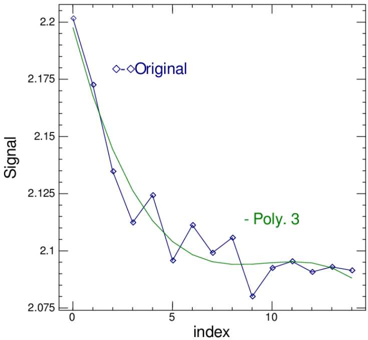


Figure : *Experimental signal (in blue) over 15 points, that we wish to adjust with a smooth curve. In green, an adjustment with a third order polynomial obtained by projecting the signal on an orthogonal base.*

Construction of an orthogonal polynomial base.

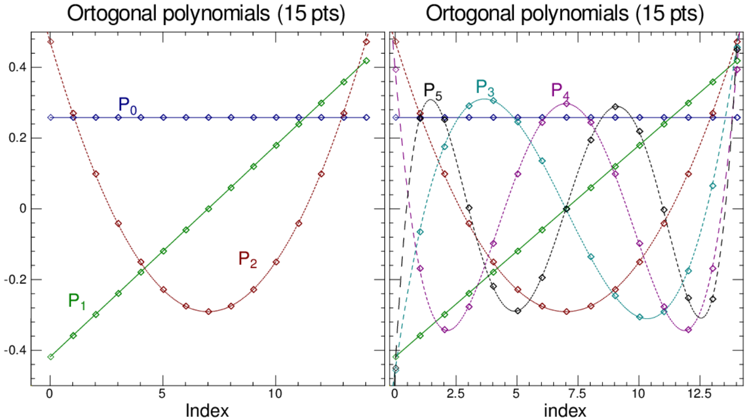


Figure : *Build-up of a base of orthogonal polynomials over 15 points. On the left, the first one P_0 corresponds to a constant value, the second P_1 is a line which mean value is null. The third P_2 is a parabola which is by definition orthogonal to P_1 and which mean value was set to zero so that it is orthogonal to P_0. On the right, we have iterated the process to define P_3, P_4 and P_5*

Discrete Fourier Transform

Signal in real space, $S_r \in C$ avec $r \in [0, N[$ The Fourier components write :

$$a_k = \frac{1}{N} \sum_{r=0}^{N-1} S_r.exp(-2.i\pi.k.r/N) \quad (1)$$

with $k \in [-N/2, N/2[$ and N points $\in \mathcal{C}$ of signals (that is $2N$ variables)$\rightarrow$ N Fourier modes $\in \mathcal{C}$ thus $2N$ variables). One can define the inverse FT :

$$S_s = \sum_{k=-N/2}^{k<N/2} a_k.exp(2i\pi.k.s/N)$$

noindent All the Fourier components content the same information that the signal in the direct space. The energy is, of course, conserved this is the **Parceval theorem**.

The Fourier modes are orthogonal

$$\frac{1}{N}\sum_{l=0}^{N-1} exp(-2i\pi.p.r/N).exp(-2.i\pi.q.r/N) = \delta(p-q)$$

. It is interesting to expand a_k in the expression of S_s :

$$S_s = \sum_{k=-N/2}^{k<N/2} \frac{1}{N}\sum_{r=0}^{N-1} S_r.exp(-2.i\pi.k.r/N).exp(2.i\pi.k.s/N)$$

One can put together and inverse the summation order :

$$S_s = \frac{1}{N}\sum_{r=0}^{N-1} S_r \sum_{k=-N/2}^{k<N/2} exp(-2.i\pi.k.(r-s)/N)$$

The sum on k the right is more easily visualized in the complex plane, it corresponds to the sum of N vectors of module 1 and which angle are at the vertex of a polygon with N sides if $r \neq s$.

Case of a real signal.

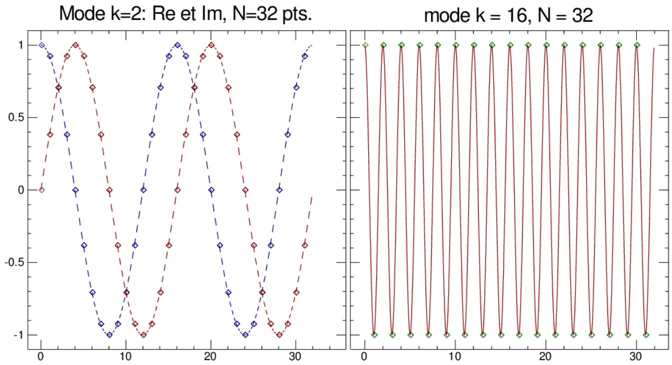


Figure : *On the left, $k = 2$ mode for $N = 32$, only the points are meaningful in view of the data sampling, the dotted lines are a guide for the eyes. In red, the cosine, in blue the sinus. On the right, $k = 16$ mode for $N = 32$. For this particular mode $k = N/2$ the real part in cosine takes only the alternating values +1 and -1, we have drawn the cosine in dashed lines as a guide for the eyes. The imaginary component in sinus (not represented) is null since the signal points are sampled exactly when the sinus is equal to zero.*

Case of a real signal.

The signal may be written $S_r \in R$ with $r \in [0, N[$ in real space, the Fourier components write :

$$Re(a_k) = \alpha \sum_{r=0}^{N-1} S_r.cos(-2.i\pi.k.r/N)$$

and

$$Im(a_k) = \alpha \sum_{r=0}^{N-1} S_r.sin(-2.i\pi.k.r/N)$$

with $k \in [0, N/2]$. The $k = 0$ mode is special since its imaginary parts is null. The $k = 1$ mode corresponds to a single oscillation of a cosine or a sinus covering the entire signal that is $[0, N]$. All modes are strictly periodic in the signal window. The $k = N/2$ mode is again very special.

Case of a real signal.

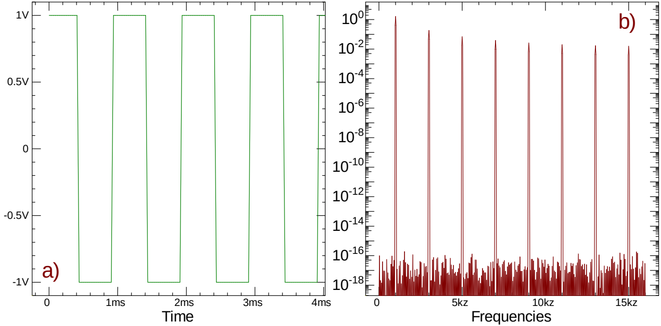


Figure : *On the left, square wave signal in real space with a period of 32 pts over* $N = 1024$*. On the right, the power spectrum in log scale. One notices that this spectrum contains only sharp peaks corresponding to*

Reconstruction of a square wave signal.

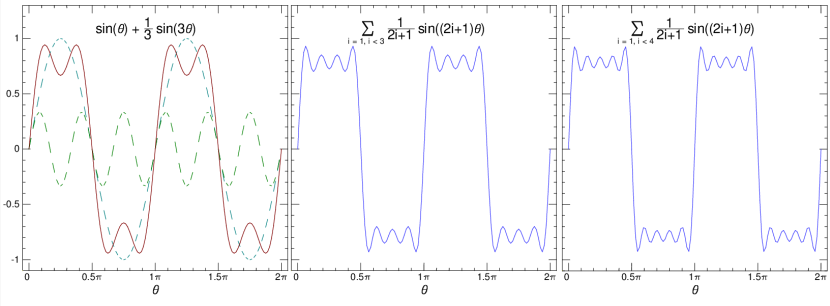


Figure : *It is possible to reconstruct a square wave signal by adding the different odd harmonics of its fundamental frequency. We present here reconstructions at various stages.*

Case of a Dirac signal

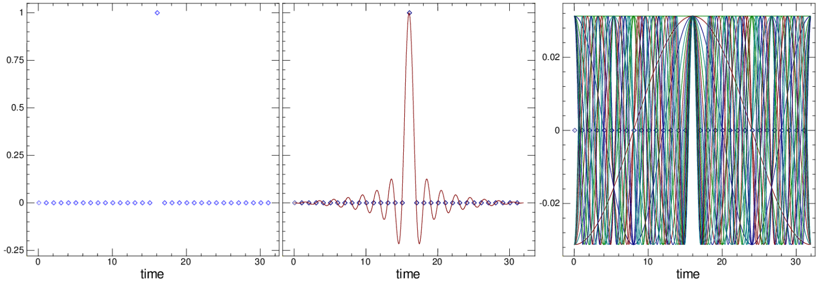


Figure : *On the left, Dirac signal in the time space with $N = 32$. On the right, ensemble of all Fourier components of this Dirac peak. In the middle, peak reconstruction using these components drawn with interpolation between the 32 initial points. In fact the Dirac peak is a cardinal sinus which maximum is located on the peak. On the other hand, the fact that the 31 remaining points in real space are null is related to the fact that these points are located precisely on the zero of the sinus cardinal.*

Periodic signals and Dirac combs

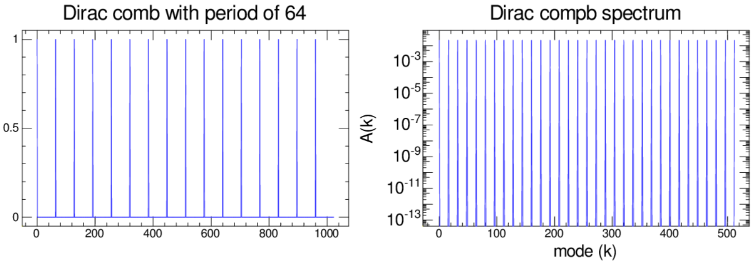


Figure : *The Fourier transform of a Dirac comb is also a Dirac comb. On the left, Dirac comb in real space with a peak every 64 points over 1024 total points. On the right, Power spectrum of the signal on the left with one peak every 16 modes.*

Relation between phases and offset (delay theorem)

$$\phi(k) = -2.\pi.k.\delta x/N,$$

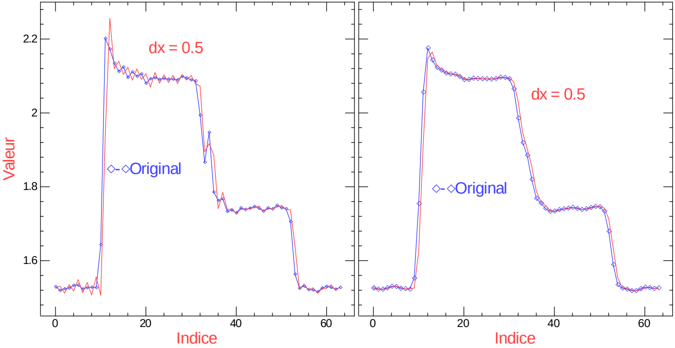


Figure : *On the left, the original signal shifted by half a pixel using the Fourier transform method. As this signal contains high frequency components, the shifted signal exhibits ringing. On the right, the shifted signal has first been low-pass filtered to suppress high frequencies components, now the shifted is smoother.*

Fourier transform and correlation function.

$$C(\tau) = \sum_{0}^{N-1} X(t).\tilde{Y}(t-\tau)$$

$$C(\tau) = \sum_{t=0}^{N-1} \sum_{k=-N/2}^{N/2-1} X_k e^{2i.\pi.k.t/N}. \sum_{k'=-N/2}^{N/2-1} \tilde{Y}_k e^{2i.\pi.k'.t/N}.e^{-2i.\pi.k'.\tau/N}$$

That is

$$C(\tau) = \sum_{k=-N/2}^{N/2-1} X_k. \sum_{k'=-N/2}^{N/2-1} \tilde{Y}'_k \sum_{t=0}^{N-1} e^{2i.\pi.(k+k').t/N}.e^{-2i.\pi.k'.\tau/N}$$

Fourier transform and correlation function.

The sum over t is non-zero only if $k = -k'$, leading to

$$C(\tau) = \sum_{k=-N/2}^{N/2-1} X_k \tilde{Y_{-k}} e^{2i.\pi.k.\tau/N}$$

Example of correlation function

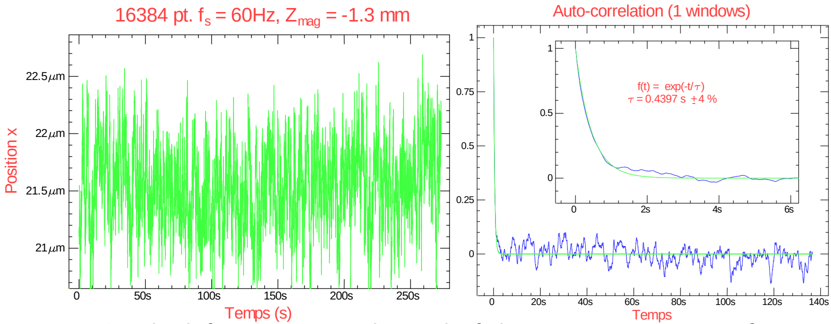


Figure : *On the left, experimental signal of the Brownian motion of a bead attached to a DNA molecule. On the right, autocorrelation function of the signal on the left (in blue), this demonstrates that the fluctuations have some memory with a characteristic time of half a second (see insert). At a longer time, the autocorrelation function presents only noise.*

Convolution product : 1D crystal

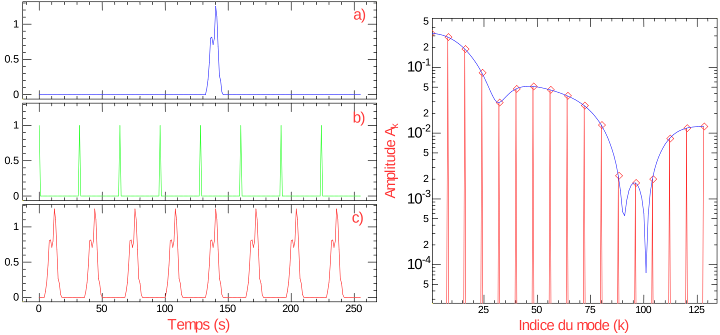


Figure : *On the left, construction of a 1D Â« crystal Â» : a) signal, b) Dirac comb, c) convolution product of a) by b) leading to a periodic signal. On the right, power spectrum of the signal a) (green continuous line), and of the periodic signal c). This signal appears as the product of the Dirac comb (not shown) by the one of the original signal. The analogy with a crystal allows better understanding X-ray pictures of crystals : the interesting information is inside one cell, one get it by inverse TF of the diffraction peaks.*

For continuous signal, the convolution product writes :

$$f \otimes g(\tau) = \int_{-\infty}^{\infty} f(t).g(\tau - t)dt$$

For a discrete signal

$$f \otimes g(\tau) = \sum_{0}^{N-1} f_r.g_{\tau - r}$$

Writing $f(r)$ and $g(\tau - r)$ in Fourier and using the delay theorem, one obtains the interesting relation :

$$f \otimes g(\tau) = \sum_{r=0}^{N-1} \sum_{k=-N/2}^{N/2-1} F_k e^{2i.\pi.k.r/N}. \sum_{k'=-N/2}^{N/2-1} G_{k'} e^{-2i.\pi.k'.r/N}.e^{2i.\pi.k'.\tau/N}$$

Convolution product and Fourier transform.

That is

$$f \otimes g(\tau) = \sum_{k=-N/2}^{N/2-1} F_k . \sum_{k'=-N/2}^{N/2-1} G'_k \sum_{r=0}^{N-1} e^{2i.\pi.(k-k').r/N} . e^{2i.\pi.k'.\tau/N}$$

Where the sum over r is not null only if $k = k'$, which leads to

$$f \otimes g(\tau) = \sum_{k=-N/2}^{N/2-1} F_k G_k e^{2i.\pi.k.\tau/N}$$

Convolution and de-convolution

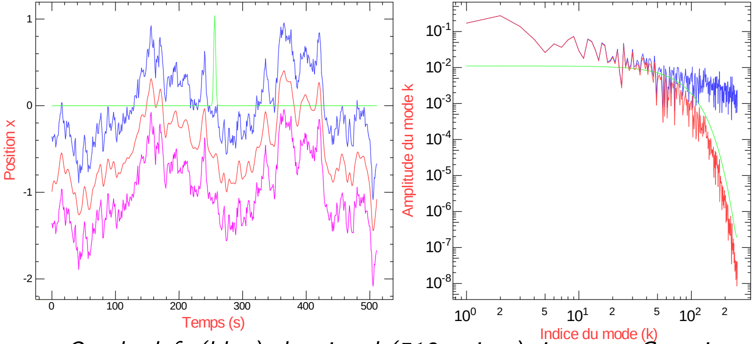


Figure : *On the left (blue) the signal (512 points), in green Gaussian response function with a width of 2 points. In red convolution of the blue signal by the green response function. In magenta, de-convolvution of the red signal by the green one using the FT. On the right, Fourier mode amplitude of the signal shown on the left with the same color coding.*

Frequency filtering

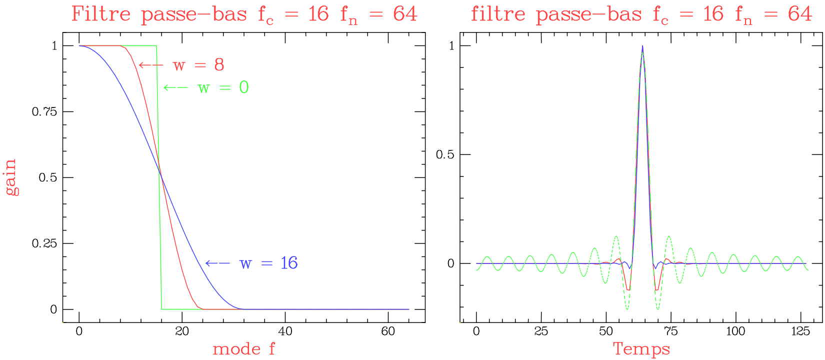


Figure : *Comparison of three low-pass filters having the same $f_c = 16$ but with different shapes. On the left, transmission coefficients in Fourier space, equals 1 for modes such tha $f < f_c - w$, null when $f > f_c + w$ and in the medium area $T = (1 + sin(\pi.(f_c - f)/2.w))/2$. On the right, impulse response of these filters in real space. The pulse width is equivalent since all filters have the same cutoff frequency at mid-height, but the decay of the ringing strongly depends on the width w of the filter.*

High-pass and pass-band filters

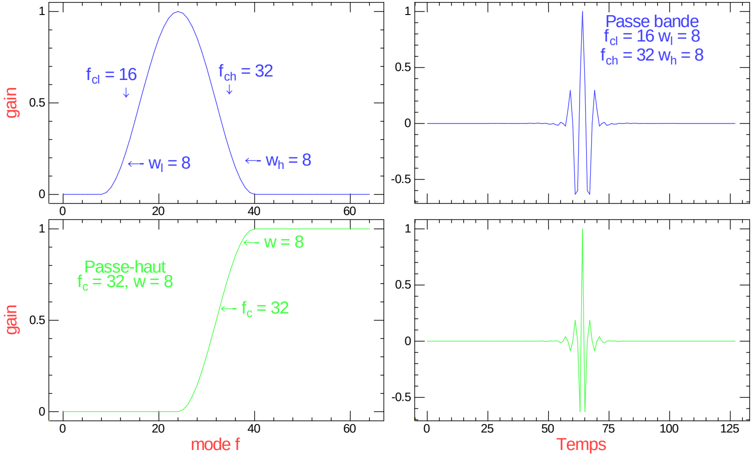


Figure : *Pass-band and high-pass filters (working over 128 points in real space). On the left, Fourier coefficient in Fourier, pass-band on the top, high-pass on the bottom. On the right, real space impulse responses of those filters .*

Non-periodic signals.

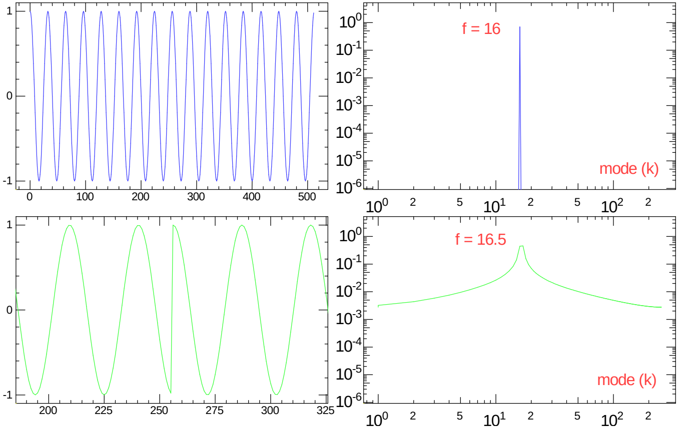


Figure : *TF of a sinus signal with a period commensurable with the analysis time window (on top $f = 16$), incommensurable ($f = 16.5$). On the left, the signal in real space has been shifted by $N/2$, for the top signal, the periodicity is perfect, for the bottom signal the shift leads to a phase jump. On the right, TF of each signal, the occurrence of a phase jump leads to strong perturbations in the spectrum.*

Hanning Window $F(r) = (1 - cos(2.\pi * r/N))/2$

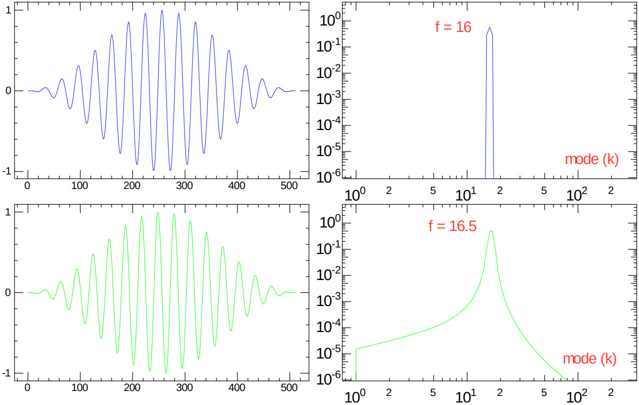


Figure : *Hanning window effect on the FT of a sinus signal commensurable with the analysis signal window (on top $f = 16$) and of an incommensurable one ($f = 16.5$). On the left, the signal in real space is multiplied by a Hanning window. On the right, FT of each signal, for the top signal, the window leads to a wider peak, for the bottom one, the phase jump effect is strongly reduced.*

Hamming window $F(r) = 0.54 - 0.46.cos(2.\pi * r/N)$

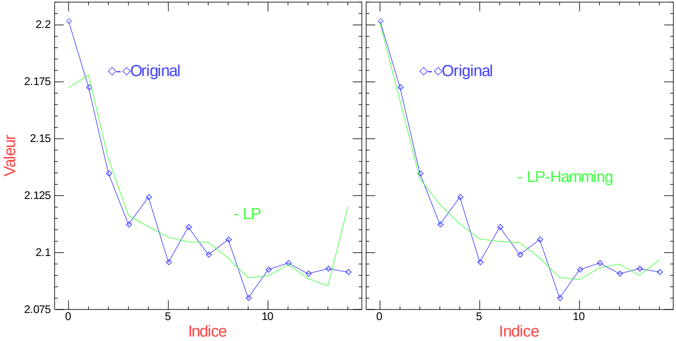


Figure : *Effect of Hamming window on filtering a signal non-periodic with the analysis window. On the left, the signal in real space treated directly with FFT. Has the signal has a discontinuity, the filtered signal presents a strong perturbation at its extremities. On the right, the signal was first multiplied by a Hamming window before low-pass filtered in Fourier space, in real space the signal was multiplied by the inverse of the Hamming window. The discontinuity at the edge has now gone.*

Low-pass filter of order 1, case of the RC circuit

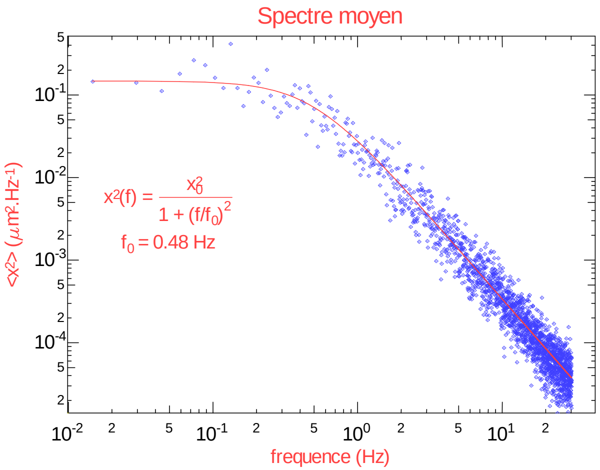


Figure : *Spectrum of the bead Brownian fluctuations tethered to a DNA molecule. This spectrum averaged four times is fitted to a Lorenzian. One needs to compare with the correlation function of figure 9since it applies to the same signal.*

Low-pass filter of second order

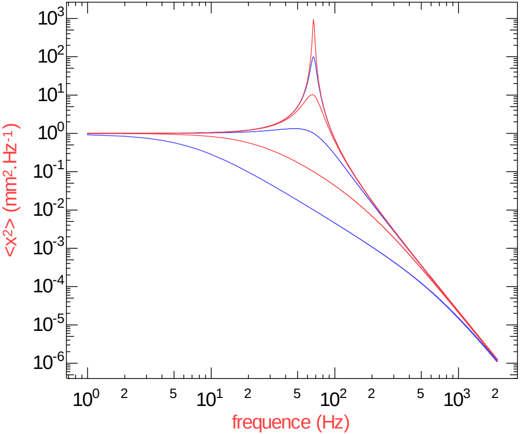


Figure : *Power spectrum of the displacement of the membrane of a loudspeaker as a function of the frequency of excitation with log-log scale. We have made the friction coefficient variable in a wide range so as to span the extreme case over high-quality resonance factor or of the over-damped situation.*

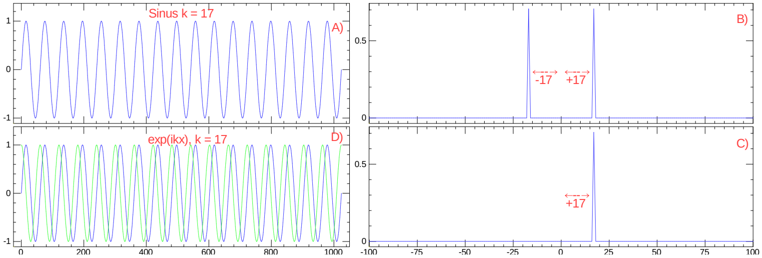


Figure : *Principle of the Hilbert transform.*

Principle of analog conversion

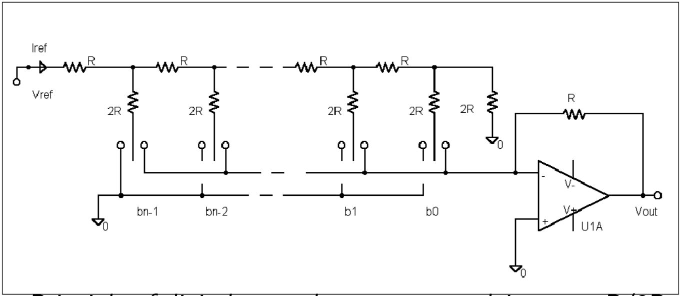


Figure : *Principle of digital to analog converter relying on a R/2R resistance network.*

Filtering before digitizing a signal, Shanon theorem

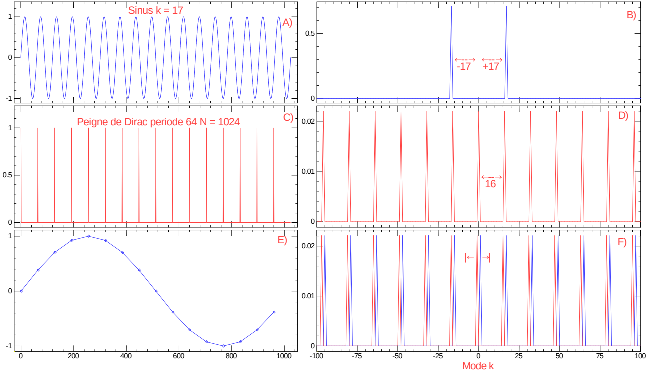


Figure : *Spectrum folding and aliasing mechanism : A) a signal of frequency $k = 17$ displayed over 1024 points. B) Fourier transform of A). C) Dirac comb corresponding to the digitaliezation with 64 points of sampling leading to 16 samples. D) FT of the Dirac comb, presenting peaks at $k = n.1024/64 = n.f_s$ with $n \in Z$ b). E) The sampled signal appears as a sinus with $k = 1$.*

They now have a few millions of pixels that act as independent sensors read synchronously. It is impossible to place a filter for each pixel before digitizing, thus video shot are prone to strong temporal aliasing artefacts. Using an integration time close to the sampling period reduces this aliasing (by filtering in averaging) but does not suppress them totally. Cameras also have spatial aliasing that is reduced by a blurring filter place just close to the pixel matrix.

Shooting a picture of an object without a lens.

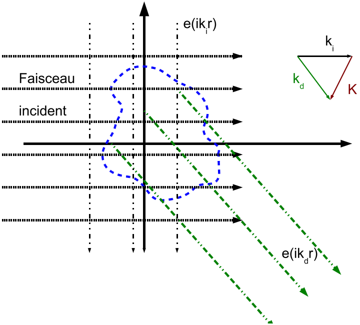
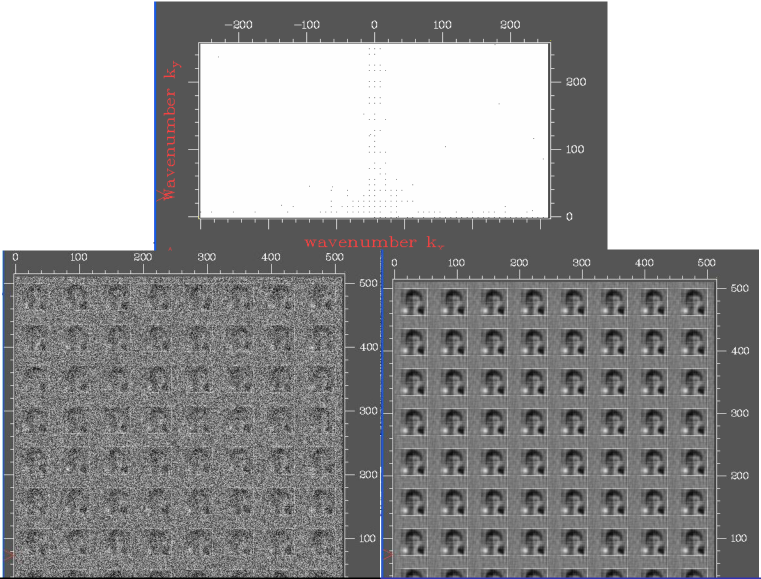


Figure : *Diffusion of a monochromatic wave by a semi-transparent object. A plane wave of wave vector $\vec{k_i}$ impact a transparent object which slightly diffuses light (in blue). The wave is diffused in different directions of space. If we consider the direction $\vec{k_d}$, the intensity of the diffused light by each point by $\vec{r}$ of the object suffers a phase shift $\phi = (\vec{k_i} - \vec{k_d}).\vec{r}$. The expression of the diffused light is nothing but the FT of the object with the wave vector $\vec{k_i} - \vec{k_d}$.*

Protein structure, the diffraction of X-ray on a crystal.



Position measurement with subpixel resolution

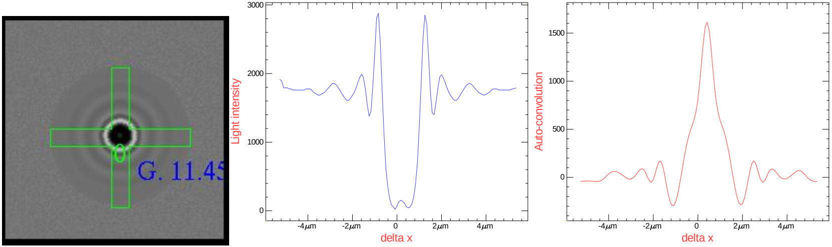
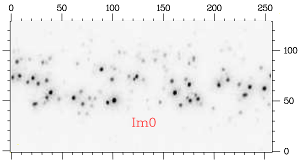
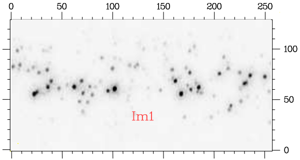
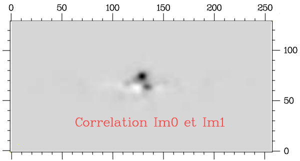


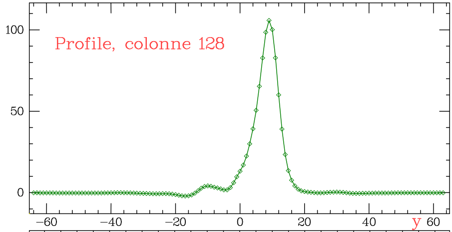
Figure : *The algorithm allowing to measure a micron size bead. On the left, the image of the bead and of a superimposed cross allowing the bead tracking. In the middle, intensity profile in X along the horizontal arm of the cross. This profile presents complex intensity modulations variable in time. The bead profile is symmetrical and we notice that it is slightly off centered on the right. On the right, auto-convolution of this profile (without its continuous part). This function present a positive maximum when δx is twice the offset of the profile with its center. By interpolating this maximum by a polynomial we can evaluate its position with an accuracy reaching the nanometer.*

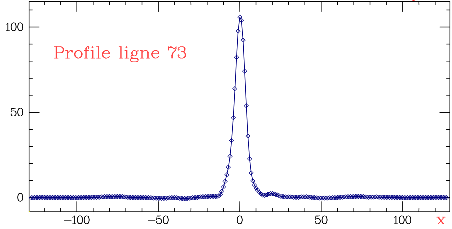
Particle Image Velocimetry PIV











Tomography

The Fourier transform of a projection is a profile in Fourier space passing by its origin. All the components of the original images along the projection direction are averaged leading a a zero width of the profile. Using the change of variable $(\vec{x}, \vec{y})$ towards $(\vec{u}, \vec{v})$ Where $\vec{v}$ is the projection direction, we can write the intensity of the projection like :

$$P(u) = \int_{-\infty}^{+\infty} \rho(u, v) dv$$

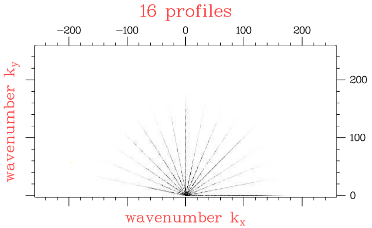
and its FT like :

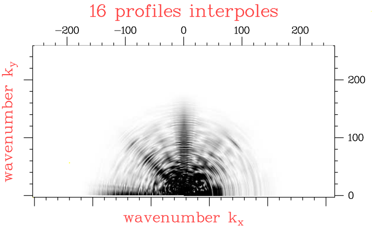
$$\tilde{P}(k_u) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \rho(u, v) e^{-ik_u.u} du.dv$$

$$= \tilde{P}(k_u, k_v = 0) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \rho(u, v) e^{-i(k_u.u+(0.v))} du.dv$$

We notice that the FT of the profile may be expressed as the 2D-FT of the image density with $k_v = 0$ and k_u whatever.

Principle of the tomography applied to a picture.





fft^{-1} 2D 16 profiles interpoles

fft^{-1} 2D 256 profiles

Original