Dynamical systems and Introduction to Chaos

23 janvier 2017

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Abstract

The goal of this chapter is to specify the conditions required for a physical system to exhibit chaotic behavior. We shall see that it is necessary that this system is nonlinear and possesses at least three degrees of freedom. A rigorous study of such a system being extremely delicate, we shall limit to a qualitative graphical approach. We shall introduce phase space and its remarkable elements. We shall introduce some preliminary characteristics essential for dynamical systems. Each one is illustrated by a simple example. We shall first introduce known regular systems to end up with simple chaotic systems. We shall discuss systems having two degrees of freedom, linear and nonlinear and then those having four degrees of freedom exhibiting chaotic behavior. We shall derive the minimal necessary conditions for a system to display chaotic behavior. Finally, we shall propose criteria allowing to characterize regular and chaotic behaviors.

1 Definitions

1.1 Physical system representation

To describe the dynamical evolution of a physical system it is convenient to use its graphical representation. Each system state is associated to a vector $\vec{X}$.Following the dynamical evolution of the system corresponds in observing the evolution of $\vec{X}$ in a vector space $\mathcal{E}$ named **phase space**. This evolution is described by an ensemble of n differential equations associated with initial conditions :

$$\frac{d\vec{X}}{dt} = F(\vec{X}) \quad (+\text{c. i.} : \vec{X}_{(t=0)}) \tag{1}$$

The application describing the time evolution of the vector $\mathcal{E}$ is called a *flow*.

1.2 Phase space and degrees of freedom

This type of vector equation describing the determinist evolution of a physical process is studied through the theory of ordinary differential equations. The vector space $\mathcal{E}$ is characterized by its dimension n. This number is also the number of degrees of freedom of the considered dynamical system.

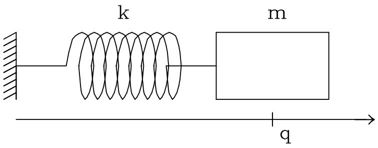

FIGURE 1 – *The harmonic oscillator is made by a mass* m *attached to a spring* k.

The concept of phase space first introduce in mechanics [1], where we find our first example of dynamical system : the harmonic oscillator in one dimension. Its position is determined by the variable q. To fully determine its state we need to specify its impulsion p. The phase space is thus of dimension two, implying that the harmonic oscillator has two degrees of freedom. This definition differs from the classical one in mechanics where a degree of freedom is associated to each couple of variables (q, p) [1]. Generally the F function may be differentiated a sufficient number of times and maybe nonlinear. In the harmonic oscillator case (see Fig.1) the function F takes the form [1] :

$$\left\{ \begin{array}{ll} \partial q/\partial t & = p/m \\ \partial p/\partial t & = -kq \end{array} \right. \tag{2}$$

1.3 Fixed points, invariant sub-spaces

The dynamic behavior of a dynamical system can be studied by characterizing its equilibrium points so-called *fixed points* :

$$\frac{d\vec{X}_0}{dt} = \vec{0} = F(\vec{X}) \quad . \tag{3}$$

The stability is studied by linearizing F near these points $\vec{X}_0$.

This leads to linear equations :

$$\frac{d\vec{X}}{dt} = \mathcal{L}_{X_0}(\vec{X}) \tag{4}$$

Where $\mathcal{L}$ is the tangential application to F en $\vec{X}_0$. Eigen values and vectors of $\mathcal{L}$ determine the time evolution of the system in the vicinity of $\vec{X}_0$.

2 Linear dynamical systems

Dynamical systems cover a broad range of situations : from mechanical systems to electronic, chemistry, thermal, etc. To illustrate the definitions that we have just given, and in particular to illustrate the various types of fixed points, we propose some simple mechanics examples.

2.1 Bead at the top of a hill

Let us now consider the case of a particle on top of a parabolic hill. This is a two degrees of freedom problems with : position q and impulsion p, related to the following equations :

$$\left\{ \begin{array}{ll} \partial q/\partial t & = p/m \\ \partial p/\partial t & = kq \end{array} \right. \tag{5}$$

The fixed point is the origin ($q = 0$, $p = 0$). Its stability is obtained by finding the Eigen values of the following matrix :

$$\begin{pmatrix} 0 & 1/m \\ k & 0 \end{pmatrix} \tag{6}$$

1. The classical form is given by a second order differential equation : $m\ddot{q} + kq = 0$. We shall prefer to use two coupled first order differential equations which correspond to the phase space variables.

The Eigen values are $\lambda = \pm\sqrt{k/m}$ and the associated Eigen vectors : $\vec{V_\pm}\begin{pmatrix} 1 \\ \pm\sqrt{km} \end{pmatrix}$. The phase space is represented in the figure 2, a trajectory corresponding to a solution of the form :

$$\vec{V} = \vec{V_+}\exp(\sqrt{k/m}\,t) + \vec{V_-}\exp(-\sqrt{k/m}\,t) \tag{7}$$

When the Eigen values verify the relation $\lambda_- < 0 < \lambda_+$, one qualifies the fixed point as **saddle point** or (*saddle node*).

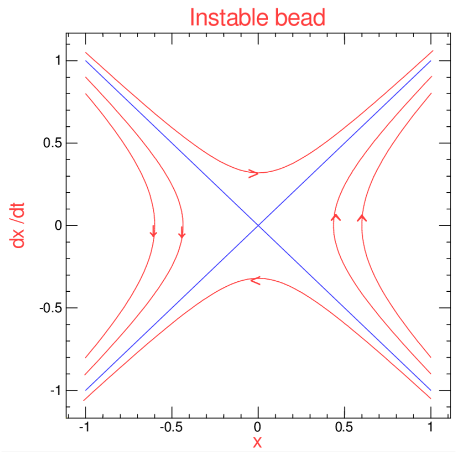

FIGURE 2 – *Phase space trajectories of a bead at the top of a hill illustrating a saddle node. They escape all to infinity, except those who initiate exactly along the contracting direction who leads exactly to the unstable point of top of the hill. The hyperboles which $x > 0$ correspond to the case where the initial speed of the bead is too small to pass the hill.*

As we could expect, the equilibrium point is unstable since one positive Eigen value exists, this means that there exists one direction in the phase space along which trajectories escape from the origin. The close trajectory corresponds to the case where the bead is launched far away from the top of the hill with an initial speed just sufficient to reach that point. The other trajectories correspond to the other cases : either the initial speed is too large and the bead pass over the hill and escapes on the other side or it is too weak and the bead turns back before reaching the hill top.

2.2 Bead at the bottom of a well

Now let us consider a bead at the bottom of a parabolic well. This system has two degrees of freedom associated with the position q and the impulsion p : this is in fact the harmonic oscillator with the following equation 2. The fixed point is still $(q = 0, p = 0)$. To study the stability of the system, let us look for the Eigen values of the matrix :

$$\begin{pmatrix} 0 & 1/m \\ -k & 0 \end{pmatrix} \tag{8}$$

The Eigen values are complex conjugates $\lambda = \pm i\sqrt{k/m}$ and the Eigen vectors : $\vec{V_\pm}\begin{pmatrix} 1 \\ \pm i\sqrt{km} \end{pmatrix}$. A trajectory corresponds to a solution having the form :

$$\vec{V} = \vec{V_+}\exp(i\sqrt{k/m}t + \phi) + \vec{V_-}\exp(-i\sqrt{k/m}t + \phi) \tag{9}$$

The phase space is represented on the figure 3. The trajectories are now ellipses. The motion is an oscillation having a frequency independent of the amplitude (the "radius" of the ellipse) given by the Eigen values $\omega_0 = \sqrt{k/m}$. As the Eigen values are complex, the fixed point is called *focus* or *center*.

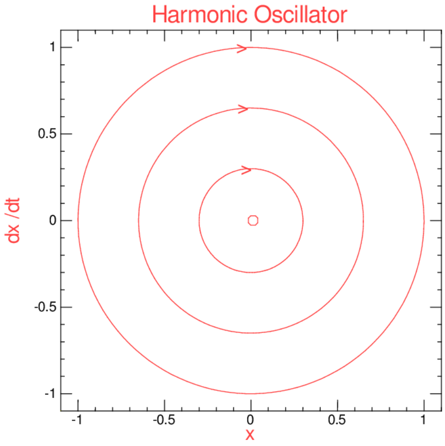

FIGURE 3 – *Trajectories in phase space of a bead in a parabolic well or a harmonic oscillator. These trajectories are traveled in the same way as the needles of a watch. The fixed point is a* center.

2.3 The different types of fixed points

For real Eigen values, the fixed point is a *stable node* if they are all negatives, a *saddle point* if some are positive. For complex Eigen values, the fixed point is a **focus** or a **center**.

3 Hamiltonian and dissipative systems

The examples that we have discussed display significative differences allowing to distinguish specific classes of systems. The two mechanical examples conserve the system energy. During the time evolution there is only conversion from potential to kinetic energy. Of course, this is only valid in the absence friction.

Hamiltonian systems or *conservative systems* correspond to systems that conserve energy and *dissipative systems* the others.

3.1 Invariant area

The phase space of Hamiltonian systems presents remarkable characteristics. On the one hand, two distinct trajectories have no common points. This is a direct consequence of the determinist character of the system : since identical initial conditions lead to the same trajectory and since all points of the phase space may be considered as initial conditions, a common point implies that the trajectories are the same. On the other hand, the Hamiltonian systems conserve areas in phase space, this is the LIOUVILLE theorem. Let us consider a volume Ω at time t_0, and let us follow its evolution in time, this volume will distort in time, but its volume will remain the same. This corresponds to the fact that the divergence of $d\vec{X}/dt$ is null :

$$\begin{aligned} \frac{dv}{dt} &= \int_{\Omega(t)} \text{div}\left(\frac{d\vec{X}}{dt}\right) dv \\ &= \int_{\Omega(t)} \sum_i \left(\frac{\partial}{\partial q_i}.\frac{\partial H}{\partial p_i} - \frac{\partial}{\partial p_i}.\frac{\partial H}{\partial q_i} \right) \end{aligned} \qquad (10)$$

where v is the volume in phase space of the domain $\Omega(t)$ it is moved around by the flow of H the system Hamiltonian. For Hamiltonian systems, this leads to an interesting property of the Eigen values of a fixed point : the sum of their real parts is always null. The bead on top of a hill falls with an Eigen value $+\sqrt{k/m}$ but it exists in a trajectory bringing back the bead to the hill top with an opposite Eigen value.

3.2 Dissipative systems

A simple way to obtain a dissipative system consists in adding friction terms to a Hamiltonian system. In the case of the harmonic oscillator, the equation becomes :

$$\begin{cases} \partial q/\partial t &= p/m \\ \partial p/\partial t &= -kq - \gamma p \end{cases} \qquad (11)$$

It is easy to see that the friction term γ induces a real part in the Eigen values $\lambda_{\pm}$ which transforms the elliptic trajectories in the case without friction to spirals converging towards the origin. It is easy to study the effect of γ on saddle points. Dissipative systems may also correspond to an increase of the system total energy : this corresponds to a negative γ value.

4 Nonlinear Hamiltonian Systems

The Hamiltonian systems have been intensively studied [2, 3], it is possible to draw the necessary conditions to obtain a chaotic trajectory. As we shall see, the nonlinear character of the equation is essential. However, all nonlinear systems do not display chaotic behavior, as we shall illustrate with the two following example :

4.1 Bead in a two well potential

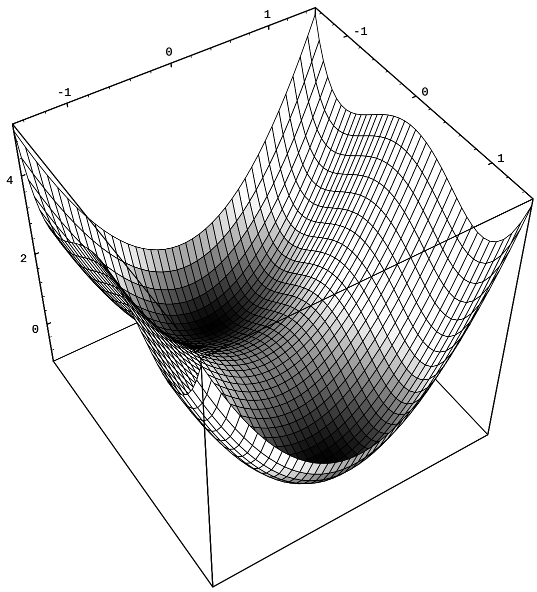

FIGURE 4 – *Representation of the surface of total energy as a function of the bead position and speed for a bead in a two well potential.*

A system is *nonlinear* as soon as its governing equations contain a nonlinear term of $\vec{X}$. This is for instance the case if the force acting on the bead writes : $\partial p/dt = q - q^3$. This situation corresponds to a bead placed in a potential having two wells, located at $q = \pm 1$. The system has two degrees of freedom and is governed by the equations :

$$\begin{cases} \partial q/\partial t &= p/m \\ \partial p/dt &= q - q^3 \end{cases} \qquad (12)$$

The exact solutions of the equation 12 are far less easy to determine than those of a linear system. We shall see later that it might be impossible to solve some nonlinear systems. Here we propose a qualitative approach to evaluate the trajectories in the phase space. We start by looking for the fixed points of the systems, there are three of them :

$$A_{-1} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \; A_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \; A_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{13}$$

Around each, we shall linearize the equations 12. Close to the origin, we recover the equations 5, thus we can say that this fixed point is a saddle node. For the two remaining ones : A_{-1} and A_1, the linear equations are those of 2. These fixed points are centers of elliptic trajectories. To obtain the trajectories in the entire phase space, one is tempted to extend *by hand* the motif formed by the saddle point and the two centers. In fact, there is a rigorous way to proceed. We have seen that Hamiltonian systems conserve energy in time. If we draw a three-dimension graph where two dimensions correspond to those of the phase space and the third one to the system total energy, we get a surface $\mathcal{S}$ with a saddle node at the origin separating two wells around each fixed point. Since the energy is invariant, a trajectory of the movement corresponds to a cut of the surface $\mathcal{S}$ at constant energy. The system trajectories are equivalent to contour level lines of the surface $\mathcal{S}$, as drawn in the figure 5.

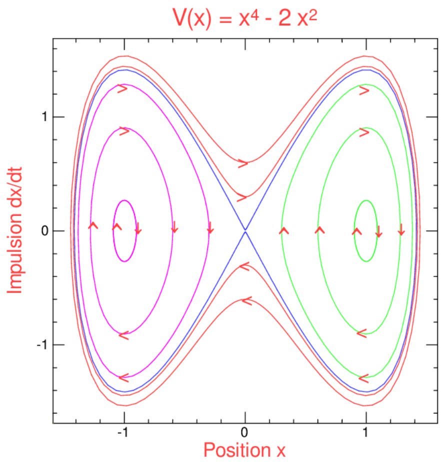

FIGURE 5 – *Phase space trajectories of a bead moving in a two wells potential corresponding to the contour lines of the surface shown in the figure 4.*

The method that we have just described is quite general : it does not depend on the nonlinear potential of the system and may be applied to all Hamiltonian systems having two degrees of freedom. As long as the surface $\mathcal{S}$ is not pathological, we shall obtain either closed trajectories or trajectories going to infinity. However, this method does not provide the actual dynamics of these movements. Still, closed trajectories correspond to periodic motions, although it is not possible to find their period by this method. Close to the center fixed points, the frequency of the movement is that of the corresponding harmonic oscillator. When we come close to a saddle point, the period increases and will diverge on reaching the saddle point.

4.2 The gravity pendulum

We will come back in many occasions to this pendulum example which constitutes one of the simplest nonlinear systems. It has two degrees of freedom (in a dynamical view) : its angular position θ and its angular velocity $\dot{\theta}$. They are governed by the equations :

$$\begin{cases} \partial\theta/\partial t &= \dot{\theta} \\ \partial\dot{\theta}dt &= -g/l \sin\theta \end{cases} \tag{14}$$

We can repeat the same analysis that we have done in 4.1. The phase space origin appears as a center point. This is the domain of small oscillations. If θ is restricted to the interval $[-\pi, \pi]$, we find two saddle nodes fixed points in $\theta = \pm\pi$ and $\dot{\theta} = 0$. The contour lines at constant energy lead to the figure 6. There exists a very peculiar trajectory joining these fixed points : it is the *separatrics*. When one follows this trajectory, the pendulum starts from the upside-down position ($\theta = -\pi$), gently separates from this point and then quickly turns around the $\theta = 0$ point and finally climbs back to reach asymptotically $\theta = \pi$. The name separatrics illustrates the fact that this trajectory is the frontier between closed trajectories corresponding to oscillations of the pendulum, with passing trajectories for which $\bar{\dot{\theta}} \neq 0$ or $\bar{\dot{\theta}}$ represent the average rotation velocity. These passing trajectories correspond to continuous rotation pendulum motions.

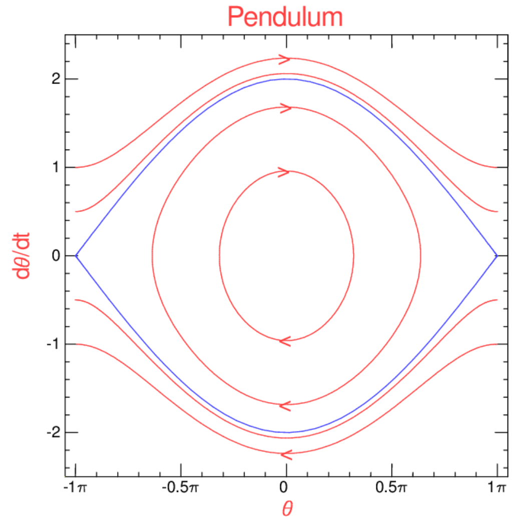

FIGURE 6 – *Trajectories in phase space of the pendulum motion. The phase space is periodic in the direction θ, upon reaching ($+\pi$, $\dot{\theta}$) the trajectories come back in ($-\pi$, $\dot{\theta}$). The trajectories called passing, correspond to the case where the pendulum has a rotating movement. The closed trajectories at the center describe oscillations of the pendulum. The trajectory which bounds these two types of motion goes from one saddle point to the next in ($\pm\pi$, O) ; It is called the* separatrics.

5 Integrability of Hamiltonian systems

A system is said *integrable* when it is possible to completely describe mathematically its trajectories in phase space. We have just seen two examples of such nonlinear systems. The fact that the energy is conserved is a characteristic property of a Hamiltonian system, allowing to determine its trajectory if it has two degrees of freedom. As a result, all nonlinear Hamiltonian systems having two degrees of freedom are integrable. This does not mean that it is easy to find the analytical solutions of the motion equations, but that they exist. Now for systems shaving a larger number of degrees of freedom n, the invariance of the energy remains, but now only tells us that the solutions belong to a space of dimensions $n-1$. But this does not allow anymore to fully characterize these solutions. However, during our mechanics lectures, we do have encountered examples of systems having more than two degrees of freedom and for which we have been able to obtain analytically their solutions. We shall recall two such examples which will allow to illustrate the necessary conditions for integrability.

5.1 System of two coupled harmonic oscillators

Let us consider two linearly coupled harmonic oscillators as sketch on figure 7.

We now need four equations to describe the system :

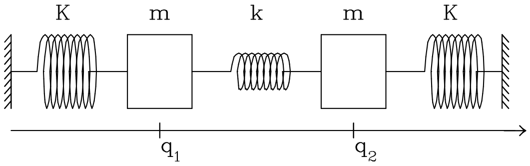

FIGURE 7 – *System of two linearly coupled harmonic oscillators.*

$$\begin{cases} \partial q_1/\partial t & = p_1/m \\ \partial p_1/dt & = -Kq_1 + k(q_1 - q_2) \\ \partial q_2/\partial t & = p_2/m \\ \partial p_2/dt & = -Kq_2 + k(q_2 - q_1) \end{cases} \tag{15}$$

As the equations are linear, we know that it is possible to find solutions. Finding them consists in finding the Eigen values and vectors of a matrix. In other words, there exist a base change where this matrix is diagonal, meaning that in this base, all the coupling between the equations disappear (here the change of variable is $u = q_1 + q_2$ and $v = q_1 - q_2$). In this new base, we can separate the coupled oscillators in new oscillators which are independent having two different frequencies : $\omega_+ = \sqrt{(K+2k)/m}$ and $\omega_- = \sqrt{K/m}$. The motion is the composition of two oscillations. We have seen that one oscillation leads to a closed trajectory in phase space (for a harmonic oscillator this is an ellipse). The combination of the two oscillations leads to a torus surface. Thus we can say that the dimension of the trajectory in phase space is two. The phase space being of dimension four, the energy invariance only restricted the trajectory dimension to three, thus an extra peculiarity of the problem restricted even further the dimension of the trajectory to two.

5.2 Two bodies problem with a central force field

The problem of two interacting bodies is a long-standing one. This is typically the *two bodies' problem* (the Earth and the Sun, for instance). As the bodies actually move along three directions of physical space, the phase space of such a system should be of dimension 12, since one needs to add three more position coordinates for the second bodies and the same number of impulsions. We shall see how it is possible to separate the different degrees of freedom and to predict the motion of this system.

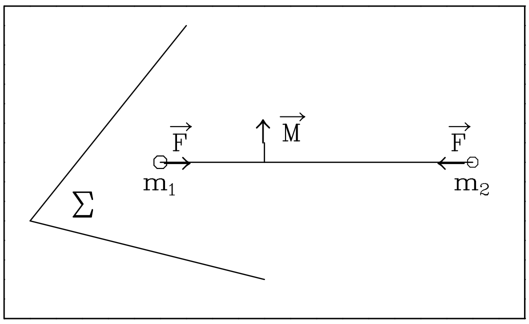

FIGURE 8 – *Two bodies in interaction with a central field force, the motion evolves in the plan Σ perpendicular to the kinetic momentum.*

Assuming that the two bodies are not submitted to other external forces and since their interaction only depends on the distance separating them, the motion of the center of mass is just a translation with a uniform speed [1], and each speed component is invariant in time. As we can associate three variables of position and three of impulsions to the center of mass, we can reduce by the same amount the number of remaining degrees of freedom by placing ourself in the center of mass referential. We now need to determine the motion of a single body submitted to a central field force, which means that the force is always directed towards the origin, and which strength only depends on the modulus of the distance to the origin r. The kinetic momentum of this particle $\vec{M} = \vec{r} \times \vec{p}$ is by definition perpendicular to $\vec{r}$ and to the force vector, and thus invariant in time. This property imposes that the motion is bound in a plane perpendicular to $\vec{M}$. We have just shown that the system may be described by two position coordinates and two impulsions. Moreover, the invariance of the kinetic momentum allows to determine the motion trajectory. Using polar coordinates (r, ϕ), the energy of the system may be written :

$$E = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + U(r) = \frac{m\dot{r}^2}{2} + \frac{M^2}{2mr^2} + U(r) \tag{16}$$

Where $U(r)$ is the interaction potential. Thanks to the kinetic momentum invariance, this energy does not depend explicitly of ϕ, and takes the form of an oscillator with an effective potential :

$$U_{eff} = U(r) + \frac{M^2}{2mr^2} \tag{17}$$

In this process, we have separated the coupled degrees of freedom in independent couples of conjugate variables (q_i, p_i). As all systems having two degrees of freedom are integrable, we know that the problem of the oscillator in the potential U_{eff} is integrable, whatever the potential form of $U(r)$. The final motion is thus the composition of an oscillation in the potential U_{eff} with a rotation at constant speed $\dot{\phi} = M/mr^2$. The oscillation is bounded between the two positive values r_{min} and r_{max}. The rotation is again a periodic behavior. Thus the trajectory in phase space is the combination of two closed curves : it has the topology of a torus. The projection of a typical trajectory in the plane (r, ϕ) is given in the figure 9.

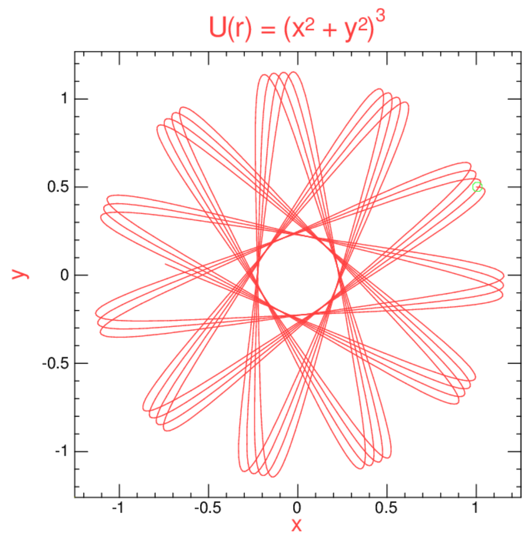

FIGURE 9 – *Trajectory of a particle in a central force field : it is the combination of a rotation and an oscillation.*

5.3 Importance of invariants, separable systems

The two examples that we have just discussed allow us to generalize the integrability concept. A system is said integrable if it is possible to break it in couples of independent conjugate variables, that is to say that it can be separated in independent oscillators. As we know how to describe the trajectories of all oscillator with two degrees of freedom (a position variable and an impulsion), the global trajectory of a system made of separable oscillators is the combination of the different oscillating behaviors. Although we can define integrability, there is no faithful method allowing to realize this separation of the system in independent oscillators. In our two examples, we have used the existence of special symmetry of the system to find the adapted change of variables leading to independent oscillators. The existence of invariants (as the kinetic momentum) is a valuable indication towards integrability. The integrability of a system having a phase space of $2n$ dimensions, implies the existence of n invariant. As a matter of fact, among the Hamiltonian systems, the trajectories of those that are integrable just explore a subspace corresponding to the composition of n oscillations, this subspace is of dimension n. For a non-integrable system, the minimal restriction corresponds to the invariance of the energy which leads to a trajectory exploring $2n-1$ dimensions of the phase space. Thus integrable systems may be viewed as peculiar systems, far less general that all systems.

6 Existence and characterization of chaos

We have just discussed the integrability concept of Hamiltonian systems, we shall now describe briefly the consequence of the absence of integrability for a system.

6.1 Two bodies in a non-central force field

What happens when the force field has no central symmetry ? First, if the force field is not anymore central, the kinetic momentum is not anymore invariant and we cannot decouple the radial oscillation from the rotation with ϕ. No need to say that solving the problem analytically to obtain the trajectory of a system becomes impossible.

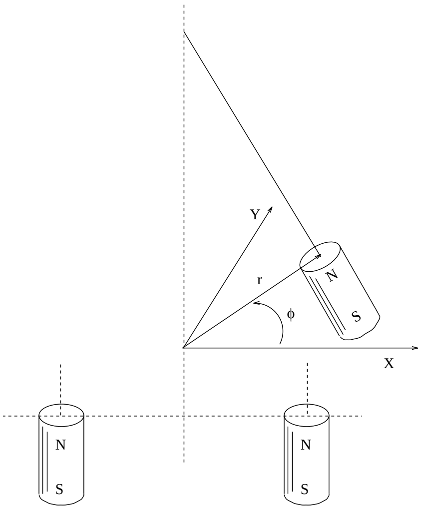

FIGURE 10 – *Magnetic pendulum device allowing to switch from a system having the central symmetry to one without this symmetry. In this last case, it allows to visualize chaotic solutions.*

The simplest way to understand what is going on is to realize a small experiment. If we attach a magnet by a thread to form a pendulum, this device might explore the direction x and y (or more exactly θ and ϕ) with the corresponding impulsion $\dot{x}$ and $\dot{y}$. We place a second magnet fixed just at the vertical of the pendulum at rest, we build a central force field system which allows recovering the trajectories of the figure 9. If now, we replace the central fix magnet by two magnets symmetrical around the origin, we create a force field that is not anymore central. Pushing the magnetic pendulum on the side leads to a special trajectory where the moving magnet is attracted now by two magnets building a double well potential. The trajectory of the pendulum will explore the two basins of attraction performing a complex but well define motion. Most of the time this trajectory is *chaotic*, a typical example is shown in the figure 11.

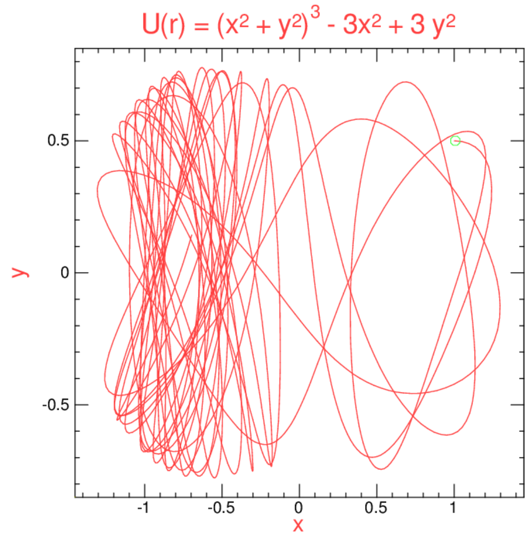

FIGURE 11 – *Trajectory of a magnetic pendulum under the influence of a non-central force field.*

6.2 Chaos characterization : power spectrum

A simple way to characterize chaos consists in performing a Fourier spectrum of the temporal evolution of one variable of the system. We have seen that the trajectories of a regular Hamiltonian system is the composition of oscillations each having a pulsation ω_i. The spectrum of such a variable contains a series of peaks located at integer values of ω_i, and to their harmonics $m\omega_i$ with $m \in \mathcal{N}$, and to linear combination of frequencies $m\omega_i + n\omega_j$ with m and $n \in \mathcal{Z}$ (see figure 12). Spectrums that are the combination of several frequencies with no simple relation are said *quasipériodic*.

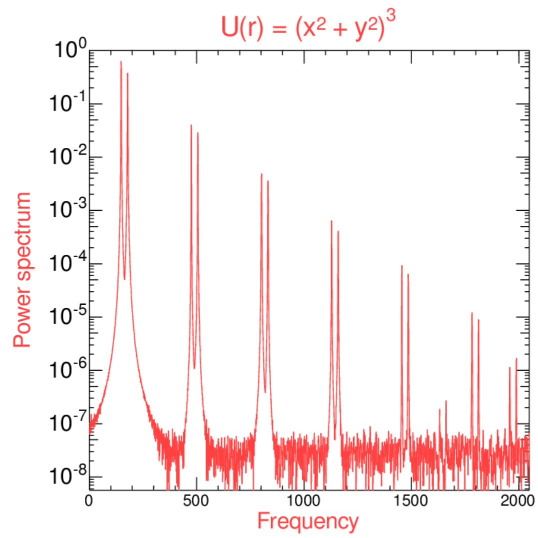

FIGURE 12 – *Power spectrum of a magnetic pendulum with a central force field, showing a regular motion. One notice the existence of two frequencies and their harmonics.*

The oscillation of the magnetic pendulum with two fixed magnets in a configuration breaking the central force field symmetry leads to a completely different spectrum without well-defined frequency peaks but rather broad band noise, as shown in the figure 13

The existence of broad band noise in the power spectrum of one variable of a system is clearly a sign characterizing chaotic behaviors. However, aside this broad band noise, such a chaotic system may also display sharp peaks in the spectrum.

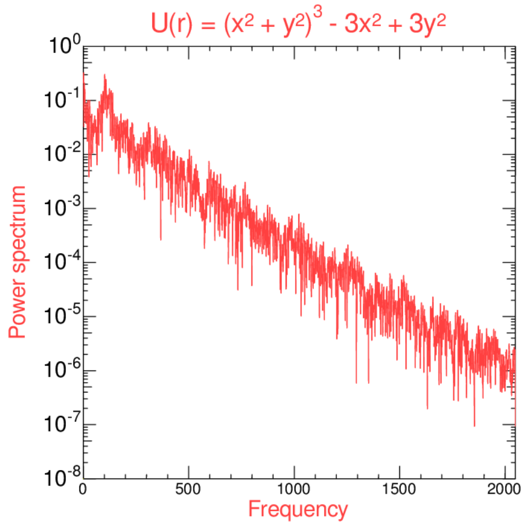

FIGURE 13 – *Power spectrum of a magnetic pendulum having a chaotic behavior.*

6.3 Characterization of Chaos : sensitivity to initial conditions

A more straightforward way to demonstrate that a trajectory is chaotic, consists in measuring it degree of non-predictability. Even though the non-symmetric magnetic pendulum is non-integrable, it is still a deterministic system. Two experiments starting from rigorously the same initial conditions, evolve with exactly the same trajectory. But if the initial conditions are not rigorously alike, the distance separating them in phase space will evolve very differently if the trajectory is regular or chaotic.

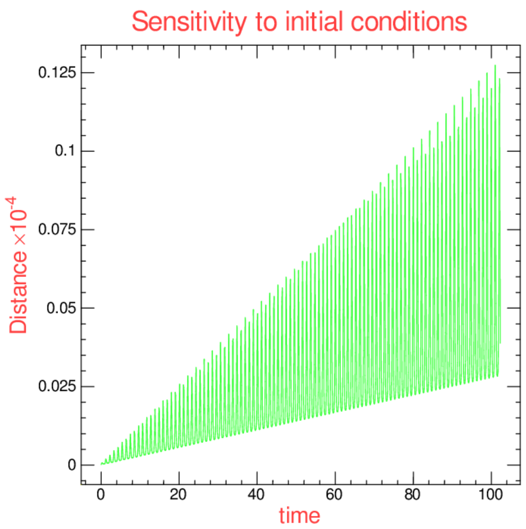

FIGURE 14 – *Time evolution of the distance between two close by trajectories for the central force field magnetic pendulum. Notice the linear scale for the distance. The fast oscillations correspond to the oscillating behavior.*

As a matter of fact, for a regular trajectory, like the magnetic pendulum under a central force field constrain, the amplitudes of oscillations, but also their frequencies, will be slightly different owing to the initial conditions. This will lead to a linear increasing phase shift between the oscillations, as two watches not perfectly tuned will do. For regular trajectories, the two systems with slightly different initial conditions will separate in phase space with a distance growing *linearly* in time on average as shown figure 14.

For a non-integrable system presenting chaotic solutions, like the dissymmetric magnetic pendulum, the oscillations arising from the two different initial conditions are well correlated at the beginning, but they quickly lead to strongly different leading to a complete loss of correlation after a few oscillations. In this case, the distance between the two trajectories increases **exponentially** with time. Notice that this evolution is not homogeneous in time, but this is only true on average. Once the trajectories are uncorrelated, the distance is bounded by the size of the trajectory as shown in the figure 15.

7 Systems with a single frequency

Let us comeback first, to the two regular examples of the particle in a double well potential and the gravity pendulum to highlight some fundamental features of non-linear systems which will become extremely important for chaotic systems. In particular, we shall discuss the occurrence of harmonics and the notion of resonance.

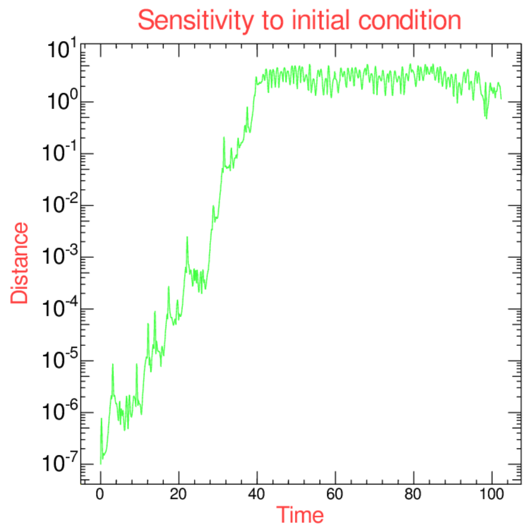

FIGURE 15 – *Time evolution of the distance between to close by chaotic trajectories. Notice the log scale. At $t = 40$, the distance saturates since the distance become comparable to the size of the explore domain in phase space.*

7.1 Small oscillation and existence of harmonics

The particle in a two wells potential allows illustrating the perturbation method for small oscillations. The force acting on the particle has the form : $f(q) = q - q^3$; For small oscillations the system remains close to the stable equilibrium points : $q = \pm 1$. Close to the point $q_+ = 1$, using the variable swap $u = q - 1$, the form of the force becomes : $f(u) = -2u - 3u^2 - u^3$. Obviously the linear term in u leads to an oscillation with the frequency $\omega_0 = \sqrt{2}$. Let us examine the non-linear terms separately u^2 u^3 and in a perturbation approach. Thus we write :

$$\left\{ \begin{array}{ll} du/dt & = \dot{u} \\ d\dot{u}/dt & = -2u - 3\epsilon u^2 \end{array} \right. \tag{18}$$

And we will look for a solution having the form :

$$u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots \tag{19}$$

Solving order by order, we obtain :

At order 0 :

lThe linear equation leads $u_0 = A\cos(\omega_0 t)$ with $\omega_0^2 = 2$.

At order 1 :

$$\left\{ \begin{array}{ll} du_1/dt & = \dot{u} \\ d\dot{u}_1/dt & = -2u_1 - 3A^2\cos^2(\omega_0 t) \end{array} \right. \quad (20)$$

By expanding the term in $\cos^2(\omega_0 t)$ under the form :

$$(A^2/2)\,[1 + \cos(2\omega_0 t)]$$

Let us decompose u_1 in two parts u_{10} et u_{12} corresponding respectively to the frequencies 0 and $2\omega_0$.

We obtain $u_{10} = -3A^2/4$, which corresponds to the shift in the mean position of oscillation, induced by the dissymmetry of the potential, relative to the transformation $+u \rightarrow -u$. Stiffer on the $u > 0$ side, the motions are easier on the $u < 0$ side.[2] The contribution at the frequency $2\omega_0$ takes the form

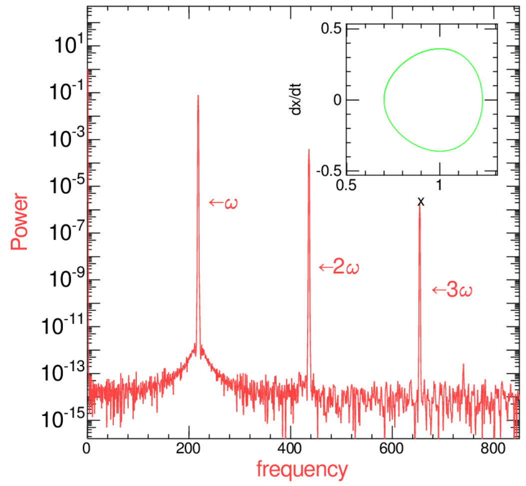

FIGURE 16 – *Power spectrum of the position of a particle oscillating in a non-linear potential. One notice the existence of harmonics of the oscillation frequency.*

$u_{12} = (A^2/4)\cos(2\omega_0 t)$ and corresponds to the occurrence of a second harmonic. Like in the case of the zero harmonic, Its amplitude varies like A^2. This justifies the perturbation approach in the limit of small oscillation amplitudes.

Let us proceed in the same way for the cubic term :

$$\left\{ \begin{array}{ll} du/dt & = \dot{u} \\ d\dot{u}/dt & = -2u - \epsilon u^3 \end{array} \right. \quad (21)$$

Order 0 leads again to the linear solution. Order 1 leads to the occurrence of a forcing term in $A^3\cos^3(\omega_0 t) = A^3/4(\cos 3\omega_0 t + 3\cos\omega_0 t)$, which corresponds to a contribution at the frequency $3\omega_0$ and to another one at frequency ω_0. If the $3\omega_0$ contribution leads to the occurrence of a third harmonic, as expected, the contribution ω_0 is precisely the linear resonance frequency of the system. But it leads to a term like $u_{31} \propto -(3A^3/4)t\sin\omega t$, which is a contribution with an amplitude growing continuously in time ! This behavior is incompatible with a perturbation approach of this computation. The failure of this method is related to the assumption that we have implicitly made on the invariance of the oscillation frequency.

7.2 Evolution of the oscillation period

To obtain a correct perturbation approach, we must assume that u is a periodic function of the frequency ω such that :

$$\omega = \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots \quad (22)$$

2. Although the zero harmonic is seldom mentioned, it definitely plays an important physical role. Just to cite one example, it is this harmonic that allows explaining the thermal expansion of solid materials linear in T or $A^2 \propto k_b T$ implying $u_0 \propto T$.[4]

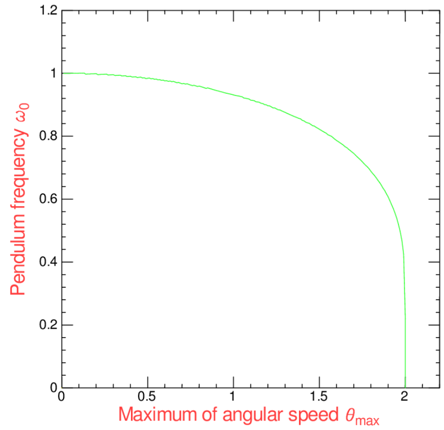

FIGURE 17 – *Evolution of the oscillation frequency of the pendulum as a function of the maximum of the rotation speed $\dot{\theta}_{max}$.*

By looking for a solution of the Looking for a solution in the form of a function of the variable $y = \omega t$ and by developing to order 1 we obtain :

$$\left\{ \begin{array}{ll} du/dy & = v \\ \omega^2 d\dot{v}/dy & = -2u - \epsilon u^3 \end{array} \right. \quad (23)$$

At order 0 :

The linear equation leading $u_0 = A\cos(\omega_0 t)$with $\omega_0^2 = 2$

At order 1 :

We obtain : $2\omega_0\omega_1\ddot{u}_0 + \omega_0^2\ddot{u}_1 = -2u_1 - (A^3/4)(\cos 3\omega_0 t + 3\cos\omega_0 t)$ (with $\ddot{u} = d^2/dy^2$). The parameter ω_1 may be chosen so that the component $\cos\omega_0 t$ disappear ; which avoids the problem that we had in the previous approach. The value of ω_1 is fixed to :

$$\omega_1 = \frac{3A^2}{8\omega_0} \quad (24)$$

Thus the oscillation frequency of the bead in a two well potential will first increase as the square of the oscillation amplitude. In the case of the pendulum, the sign of the term in θ^3 is different, and one finds $\omega_1 = -A^2/(8\omega_0)$: its frequency decreases with the amplitude. The perturbation approach that we have sketched demonstrates the occurrence of harmonics directly related to the non-linear terms and also the importance of the frequency variation as a function of the amplitude. In the particular case of Hamiltonian systems having a phase space of dimension two, it is possible to write the expression of the oscillation period in an integral form of the system total energy :

$$\frac{m\dot{q}^2}{2} + U(q) = E \quad (25)$$

Leading to :

$$t = \sqrt{2m}\int_{q_{min}}^{q_{max}} \frac{dq}{\sqrt{E - U(q)}} \quad (26)$$

where q_{min} and q_{max} correspond to the extreme excursions of the oscillation. Notice that the integral of the equation 26 is elliptical. We have reproduced the typical evolution of the oscillation frequency of the pendulum versus the maximum value of the rotation speed on the figure 17.

7.3 Structure of a resonance

One convenient model of the gravity pendulum studied previously is a compass in a magnetic field. The two systems are completely equivalent but in the second one, it is extremely easy

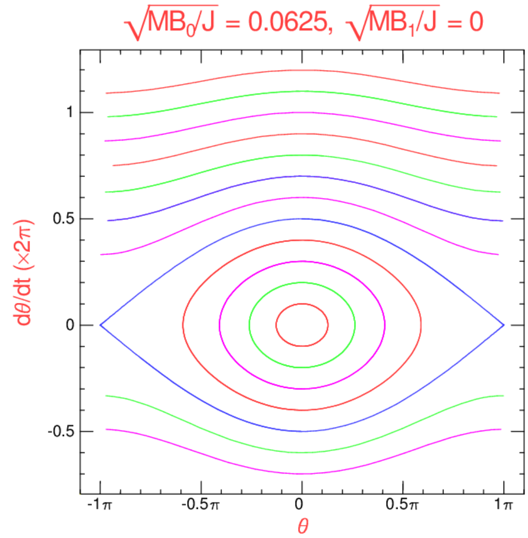

FIGURE 18 – *Trajectory of a compass in a magnetic field or of the angle of a gravity pendulum in the phase space. The oscillation domain is bounded by the separatrics which defines a* resonance.

to modulate the potential by changing the magnetic field. If we call $\vec{\mathcal{M}}$ the compass magnetization, J its inertial momentum and $\vec{B}$ the magnetic field applied. The equation describing the system writes :

$$\left\{ \begin{array}{lcl} d\theta/dt & = & \dot\theta \\ d\dot\theta/dt & = & -(\mathcal{M}B/J)\sin\theta \end{array} \right. \quad (27)$$

If the magnetic field applied is null, $B = 0$, the trajectories in phase space are simply straight lines parallel to the axis $\theta = 0$ since $\dot\theta = cte.$. The motion of the compass is a uniform rotation in one direction or the other. Let us now observe the evolution of the trajectories when we increase the magnetic field to the value B_0. As we have drawn on the figure 18, trajectories of *center* type emerge around the origin and saddle points appear in $\theta = \pm\pi$ and $\dot\theta = 0$. The oscillation domain is bounded by the special trajectory which is the *separatrics*, already described in the pendulum case. The trajectories of rotation at constant speed are modified and their rotation speed is now modulated. So to speak the magnetic field B_0 as pushed the rotating trajectories to insert a set of oscillation trajectories. This island of oscillating trajectories is called a resonance. It is easy to compute the extension in phase space of this island : As a matter of fact, its half-width corresponds to the maximum speed that the compass acquire in following the separatrics, namely :

$$\dot\theta_{max} = 2\sqrt{\mathcal{M}B/J} \quad (28)$$

The half-width of the resonance increases with the magnetic field B. Since $\dot\theta$ is also the pulsation, it is interesting to compare it to the small oscillation frequency : $\omega_0 = \sqrt{\mathcal{M}B/J}$. One finds that $\dot\theta_{max} = 2\omega_0$.

8 Systems with two frequencies

We now propose to study qualitatively the transition to chaos of simple Hamiltonian systems. We have seen that it is enough that the phase space has four dimensions for a system to become non-integrable. The compass in a magnetic field provides a control parameter that we shall use to drive the system from an integrable situation towards a non-integrable one. We shall also show that one dimension of the phase space corresponds to a trivial behavior ; thus it is possible to describe the evolution in a phase space easier to represent by having only three dimensions.

8.1 Compass subject to two magnetic field

Let us imagine that we place a compass in a static magnetic field B_0 and that we approach this compass by a second compass so that a coupling exists between the two. We have just built a system having a phase space with four dimensions since we need to use the variables θ and $\dot\theta$ to describe the first compass and ϕ $\dot\phi$ for the second one. This system is non-linear and has enough degrees of freedom to present chaotic trajectories.

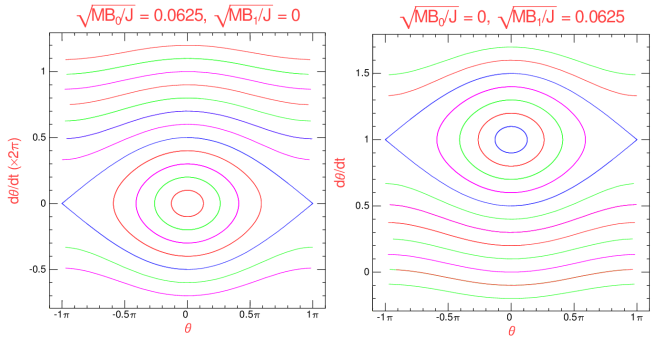

FIGURE 19 – *In the limit cases where one of the magnetic field is null, the phase space corresponds to the occurrence of one resonance centered on $\dot\theta = 0$ when $B_1 = 0$ (on the left) and centered on $\dot\theta = 1$ in the case $B_0 = 0$ (on the right).*

Let us now assume that the second compass is far much bigger and heavier than the first one : Its kinetic momentum J_ϕ is far greater than the one of the first compass. As a matter of fact, the motion of the small compass will have a limited effect on the second, while the reverse is not true. In the limit where the ratio of the kinetic momentum goes towards infinity, the motions of the second compass are not altered by those of the first one. As a result, the second compass may be viewed as evolving alone in a two-dimensional phase space and to have a specific invariant $\dot\phi = \Omega$. This second invariant replaces the energy one in an interesting manner since it allows considering only a three-dimension phase space : θ, $\dot\theta$ and ϕ. On a practical point of view, the effect of the big compass, may be summarized as imposing a rotating magnetic field with the frequency Ω. A way to build such a system, consists in placing a compass simultaneously in a static field B_0 and in a rotating one B_1. If we write $\mathcal{M}$ the magnetization of the compass and J its kinetic momentum, the equations governing the system take the form :

$$\left\{ \begin{array}{lcl} d\theta/dt & = & \dot\theta \\ d\dot\theta/dt & = & -\frac{\mathcal{M}B_0}{J}\sin\theta - \frac{\mathcal{M}B_1}{J}\sin(\theta-\phi) \\ d\phi/dt & = & \Omega \end{array} \right. \quad (29)$$

It is worth noting that if we shut down the static field keeping only the rotating on ($B_0 = 0$ and $B_1 \neq 0$), we recover the problem of a compass in a static field (that is the one of the gravity pendulum) after performing a change of referential by placing ourselves in a rotating space $\theta_1 = \theta - \Omega t$ having the same rotation speed as the magnetic field. In this new referential the rotating field appears static and it leads to the occurrence of a resonance which is just shifted along the axis $\dot\theta$ by the quantity Ω. If the two fields are non zero, each field leads to a resonance : this system is called a two resonances system.

8.2 Stochasticity criterion

In the case of this compass, a resonance corresponds to a domain in phase space where the oscillation motions occur around one of the fields. A resonance may be viewed as a *capturing zone*.

Thus, it is impossible for a trajectory to belong to both resonances simultaneously, this would mean that the motion of the compass arises as an oscillation around the static field and also around the rotating one ! As the size of the resonance increases with the strength of the associated field, it is clear that if we increase the strength of the field a problem will arise when the two resonance start to overlap. In this situation, the compass does not know which field to follow, and it generally adopts a chaotic behavior, following alternatively each field in a random manner. On the contrary, one imagines that when the field amplitude is small, the compass motion will be regular. This simplistic reasoning allows obtaining a rudimentary criterion to predict the occurrence of chaotic behaviors : the overlapping of the resonance takes place when the sum of the half-widths of the resonances equals the distance separating them. This is the **Stochasticity criterion**.

$$\mathcal{S} = \frac{2\sqrt{\mathcal{M}B_0/J} + 2\sqrt{\mathcal{M}B_1/J}}{\Omega} = \frac{2(\omega_0 + \omega_1)}{\Omega} \quad (30)$$

where ω_0 and ω_1 are respectively the small amplitude oscillation frequencies of the compass in the fixed and the rotating field respectively. We see that the resonance overlap occurs when $\mathcal{S} = 1$, which happens either when we increase the field amplitudes or when the frequency of the rotating field decreases Ω. This criterion can apply to different systems, where there exists resonances separated along the $\dot{\theta}$.

9 POINCARÉ section

Until now, we have been careful not to try representing the three-dimensional phase space of the compass. We have also used for these systems the resonance concept introduce for the phase space of systems having two dimensions, without justifying the validity of the approach for chaotic trajectories. In fact, by introducing a more appropriate representation of the phase space, we will see that the resonances globally exist in the compass system but that they are suffering some important perturbations.

9.1 Trajectories stroboscopy

The compass may be seen as a system perturbed by a signal at frequency Ω. One classical way to check that the compass oscillates or turns regularly with this frequency, consists in observing it with a stroboscope. That is to say, to observe its position and speed at periodic time corresponding to $t_n = n2\pi/\Omega$. This is equivalent as performing a section of the trajectory of the compass in a plane $\phi =$ cte. Instead of observing a continuous trajectory, we now have a series of intersection points in the plane $\theta, \dot{\theta}$: This is a POINCARÉ *section* (we have used the periodicity of ϕ to collapse the trajectory in the interval $[0, 2\pi]$). The POINCARÉ section is a very useful tool allowing to reduce the phase space dimension by one.

In the case of the compass, we bring back the study to the plane $\theta, \dot{\theta}$, similar to the one used for the gravity pendulum. However now, the trajectory is not anymore a continuous curve, but a series of discrete points (in principle one should index them according to their apparition number but this is seldom done).

9.2 POINCARÉ section

Although the trajectory of a given dynamical system does not always present a specific periodicity imposed externally, it is always possible to realize a POINCARÉ section as the ensemble of the intersection points of this trajectory why a given plane of the phase space. One only keeps the points intersecting the plane with a given direction. The time separating two intersections is then not anymore constant. A POINCARÉ section reduces the information contained by the complete trajectory, it might even be that it does not represent the entire dynamics. To grasp a realistic view of the system, it might be necessary to vary the section plane (either the phase of the stroboscopy or its direction).

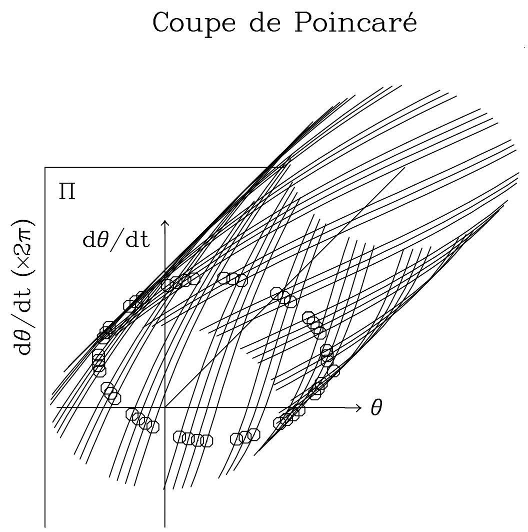

FIGURE 20 – *Principle of a* POINCARÉ *section of the compass.*

9.3 Typical POINCARÉ section of the compass

To illustrate this concept, let us observe the section shown in the figure 21, done at $\phi = 0$, with equal magnitude of both magnetic field, and at $\mathcal{S} = 1/2$.

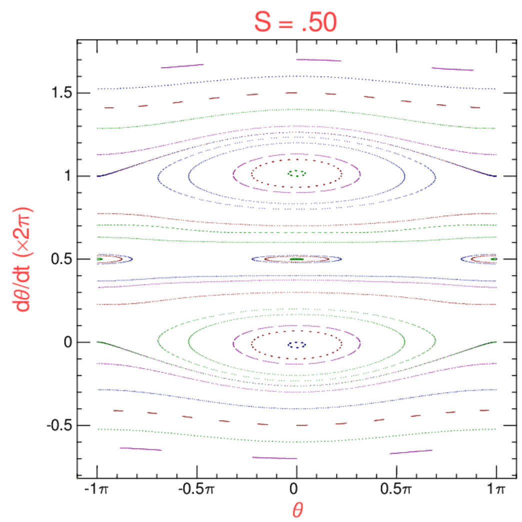

FIGURE 21 – *Compass* POINCARÉ *section at* $\mathcal{S} = 1/2$. *By varying the initial conditions, it is possible to distinguish different regions of the phase space, in particular the resonances.*

For some peculiar initial conditions, the point series align along well-defined curves. One recover the two resonances and the rotation trajectories.

10 Passing trajectories

We are going to describe the evolution of the rotation trajectories located in between the two principal resonances. It is easy to imagine that passing trajectories are not much impacted by the resonances as long as $\mathcal{S} << 1$, but what occurs when $\mathcal{S}$ comes close to 1 ? This picture is qualitatively correct, however, the trajectory is altered by the occurrence of frequency lock-ins leading to a special structure of the phase space. We will try to compute

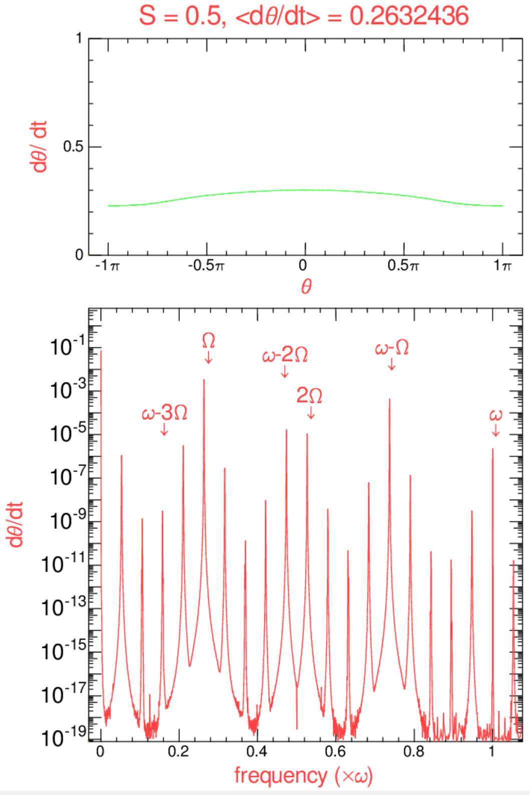

FIGURE 22 – *Analysis of the passing trajectories. At the top Poincacé section, one notices that the trajectory is dense. At the bottom, power spectrum of the angular velocity of the compass.*

these trajectories using a form developed in ϵ as we have done already. We assume that B_0 and B_1 to be proportional to ϵ (and thus small). We seek to solve :

$$\ddot{\theta} = -\epsilon M \sin\theta - \epsilon P \sin(\theta - \Omega t) \tag{31}$$

Where $M = \mathcal{M}B_0/J$ and $P = \mathcal{M}B_1/J$. We are looking for a solution having the form :

$$\theta(t) = \theta_0(t) + \epsilon\theta_1(t) + \epsilon^2\theta_2(t) + \cdots \tag{32}$$

At order 0 : we need to solve : $\ddot{\theta}_0 = 0$, leading to :

$$\boxed{\theta_0(t) = \phi + \omega t + \cdots}$$

This is a trajectory with a constant rotating speed ω ; the phase term ϕ corresponds to the multiple possibilities of the phase value. **At order 1 :** we need to solve :

$$\ddot{\theta_1} = -M\sin(\phi + \omega t) - P\sin\left(\phi + (\omega - \Omega)t\right)$$

We obtain :

$$\boxed{\theta_1(t) = \frac{M}{\omega^2}\sin(\theta_0(t)) + \frac{P}{\omega'^2}\sin({\theta'}_0(t))}$$

Where ${\theta'}_0(t) = (\phi + (\omega - \Omega)t)$. While rotating, the compass « feels » the fixed field as a periodic perturbation at frequency ω. Its rotation speed is modulated by this frequency. In a similar fashion, the rotating field also imposes a periodic modulation at frequency $\omega' = \omega - \Omega$. These two perturbations can be seen in the power spectrum of the compass which now contains two peaks at the frequencies : ω and ω'.

At order 2 : we need to solve :

$$\ddot{\theta} = -\epsilon M \sin\left(\theta_0(t) + \epsilon\theta_1(t)\right) - \epsilon P \sin\left({\theta'}_0(t) + \epsilon\theta_1(t)\right)$$

By expanding the sinus and keeping the terms in ϵ^2 we get :

$$\begin{aligned}\ddot{\theta_2} = \quad & -M\cos\theta_0\left[\frac{M}{\omega^2}\sin\theta_0 + \frac{P}{\omega'^2}\sin{\theta'}_0\right]\\ & -M\cos{\theta'}_0\left[\frac{M}{\omega^2}\sin\theta_0 + \frac{P}{\omega'^2}\sin{\theta'}_0\right]\end{aligned}$$

The term $\cos\theta_0\sin\theta_0$ leads to a perturbation in $\sin(2\phi + 2\omega t)$ this is the second harmonic at 2ω ; In a similar fashion the term $\cos{\theta'}_0\sin{\theta'}_0$ leads to the second harmonics of $2\omega'$. The mixed terms $\cos\theta_0\sin{\theta'}_0$ et $\cos{\theta'}_0\sin\theta_0$ lead to terms $\theta_0 \pm {\theta'}_0$ and ${\theta'}_0 \pm \theta_0$ which correspond to perturbations at frequencies $\omega \pm \omega'$ and $\omega' \pm \omega$. In general we can determine the expression of θ_2 and pursue the expansion to higher order. However, in the case where $\omega = \Omega/2 = -\omega'$, we have to solve the equation : $\ddot{\theta_2} \sim -\sin(2\phi)$. Which again lead to a term growing like t^2 which does not become small contrary to our assumption made at the beginning of this calculation in 32. In the case where $\omega = \Omega/2$ we have a frequency lock-in.

10.1 Frequency lock-in 1/2

When two oscillators interfere they usually lock-in phase, it is easy to understand that the small phase difference matters. In the 1/2 lock-in example, we have just shown that the compass is under the influence of a perturbation at $\sin(2\phi)$. To correctly describe this lock-in one needs to start again the perturbation computation, but this time in allowing the frequency ω to evolve as we had done in 7.2. More precisely, as a frequency modulation is just a phase modulation, we are going to assume that ϕ varies in time. We shall look for a solution under the form :

$$\theta = \frac{\Omega}{2}t + \phi(t) + \cdots$$

We shall analyze the corresponding motion associated to this lock-in in a particular case. We choose $M = P$, the equation 31 becomes :

$$\ddot{\phi} = -2\epsilon M \sin\phi\cos(\Omega t/2)$$

Let us notice that in this situation, the point corresponding to à $\phi = 0$ and $\dot{\theta} = \Omega/2$ is a fixed point as well as the point corresponding to $\phi = \pi$. To appreciate the solution, we shall assume that the phase ϕ is small and close to 0 but can oscillate slowly around that fixed point :

$$\phi = \exp(\epsilon\omega_0 t)[u_0(t) + \epsilon u_1(t) + \epsilon^2 u_2(t) + \cdots]$$

At order 0 : one obtains : $\boxed{\ddot{u}_0 = 0}$
At order 1 : we need to solve :

$$\ddot{u}_1 - i\omega_0\dot{u}_0 = -2Mu_0\cos\left(\frac{\Omega t}{2}\right) \tag{33}$$

This leads to :

$$\boxed{\dot{u}_0 = 0} \text{ et } \boxed{u_1 = \frac{8M}{\Omega^2}u_0\cos\left(\frac{\Omega t}{2}\right)}$$

At order 2 : we need to solve :

$$\ddot{u}_2 - i\omega_0\dot{u}_1 - \omega_0^2 u_0 = -2Mu_1\cos\left(\frac{\Omega}{2}t\right)$$

By replacing u_1 by its expression 33, we separately solve the terms that explicitly depend on time, those in $\cos(\Omega t)$ and those in $\cos(\Omega/2t)$. This allows to find the expression of ω_0 :

$$\boxed{\omega_0^2 = \frac{8M^2}{\Omega^2}}$$

Thus we see that the oscillation frequency of the phase is well-defined and is equal to $\epsilon\omega_0$. Near the fixed points $\theta = 0, \dot{\theta} = 1/2$ et $\theta = \pm\pi, \dot{\theta} = 1/2$, we observe a phase oscillation that corresponds to the elliptical trajectories visible in the figure 21. The equations in $\cos(\Omega t)$ and $\cos(\Omega/2t)$ allow describing the expression of u_2 :

$$\begin{aligned} u_{21} &= i\omega_0 \frac{16M}{\Omega^3} u_0 \cos\left(\frac{\Omega t}{2}\right) \\ u_{22} &= -u_0 \frac{32M^2}{\Omega^4} \cos(\Omega t) \end{aligned} \tag{34}$$

10.2 The spectrum of the rotation motion

We can generalize the results of the expansion in ϵ by considering the frequency spectrum of the rotation speed of the compass : As long as the periodic perturbations do not have a zero frequency, the higher orders in ϵ lead to frequencies like $p\omega \pm q\omega'$ with p and $q \in \mathcal{Z}$. This is a complex spectrum but all the peaks are in fact frequency combinations of ω and ω' that defines the *rotation number* $\sigma = \omega/\omega'$.

10.3 Rational rotation number

Each time this rotation number σ is a rational equal to p/q, the ϵ expansion is singular : A *frequency lock-in* appears. The rotation trajectory then breaks to a series of q small resonances, these q resonances lead to the occurrence of q center points as well as q saddle points. One can observe an example of this behavior in figure 21. Upon the frequency lock-in occurrence, one might think that that the two starting frequencies ω and ω' by combining in a rational manner will lead to a system with only a single remaining frequency. But as we have seen for the $1/2$ lock-in, the phase oscillation brings in a new frequency $\epsilon\omega_0$ leading again to a two frequencies system. This new frequency may again present a commensurate relation with the fundamental frequency, a lock-in and the birth of a new frequency. In general, the system still present two frequencies. The amplitude of the series of resonances occurring upon such a lock-in, is stronger if the rationale is simple, that is if q is small. At equal distance from the fundamental resonances, one observes a series of resonance corresponding to the lock-in $1/2$. Those of the $1/3$ lock-in are weaker and so on.

10.4 K.A.M. torus

If the ratio $\sigma = \omega/(\Omega - \omega)$ is an irrational number, the lock-in mechanism seen previously, does not occur and the expansion ϵ in can be constructed. However, it is possible that this expansion does not converge. The trajectory remains as long as the perturbation related by to resonances are not too strong. While introducing the stochasticity criterion, $\mathcal{S}$ we have shown how the fundamental resonances could perturb the passing trajectories. The occurrence of the series of secondary resonances will also destabilize the irrational nearby trajectories. In other words, some trajectories will remain regular even though the system is non-integrable. This proposition constitutes the theorem Kolmogorov Arnold and Moser (K.A.M.). Its demonstration was a master piece of Mathematics from these three authors. Its implication is capital since it allows stating that some passing trajectories located in between two fundamental resonances will remain rigorously regular until a finite threshold in stochastic parameters will be reached. This type of trajectory corresponds to a series of points forming a well-defined curve as in figure 21. This theorem allows justifying the expansion in ϵ that we have sketched. In principle, it allows specifying the ϵ parameter domain for regular trajectories.

11 Trajectories inside a resonance

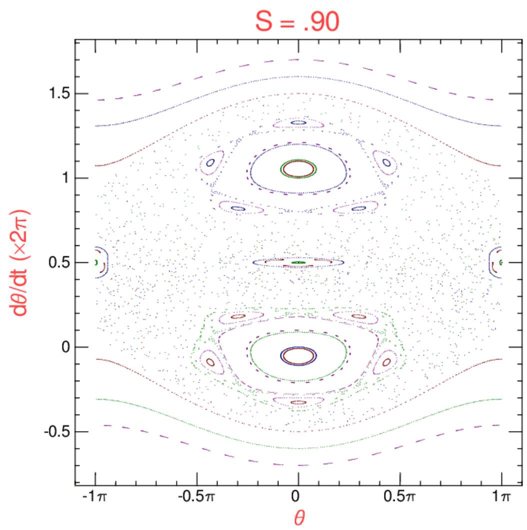

FIGURE 23 – POINCARÉ *section of the compass at* $\mathcal{S} = .9$. *One notices the occurrence of five islands corresponding to a lock-in* $1/5$.

We have just seen that the ratio σ of the two frequencies characterizing a passing trajectory, determines its stability, at least for weak stochasticity parameter values. As this ratio changes continuously when the angular speed increases, near the resonance center this ratio is maximum at Ω and decreases towards 0 as the amplitude of the oscillation increases towards the resonance separatrics. During that evolution, the ratio will alternate between rational and irrational leading to a very complex structure. Let us examine this situation around the mean strongest lock-in. Let us consider the case of the resonance associated with the fixed field. At first order, the compass oscillates around the fix field with a frequency ω. This oscillation suffers the perturbation of the rotating field that induces a second oscillation at frequency Ω. We are facing a two frequencies system, equivalent to the case of the passing trajectory that we have just studied. In a similar manner, the stability of the trajectories will depend of the rational character of the ratio $\sigma = \omega/\Omega$.

11.1 Secondary resonances

This is at this stage that the non-linear behavior of the compass oscillation takes all its importance. As shown in the figure 17, the frequency evolves continuously from the small oscillation frequency ω_I towards 0. The situation is comparable to that of the passing trajectories. However, the linear oscillation area is singular owing to the second order variation of the frequency ω with $\omega \approx \omega_l$. One observes a mixture of lock-ins and regular trajectories associated with KAM tori. The lock-ins lead to occurrence of series of periodic islands as those of the figure 23.

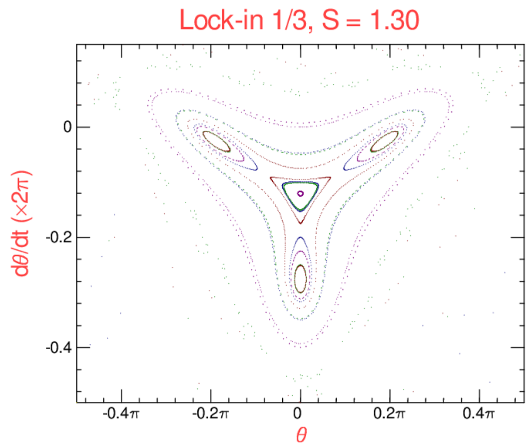

FIGURE 24 – POINCARÉ *section of the compass at* $\mathcal{S} = 1.30$. *Lock-in 1/3, notice that the saddle points and the center points are not anymore aligned with an ellipse.*

In this domain where the oscillations are nearly linear, the amplitude of the harmonics is small and the lock-in are very weak. On the other hand, as we go away from the resonance center, the lock-in increase in strength and secondary islands becomes larger. The various lock-ins have their frequency ratio ω/Ω bounded by $\omega_l/\Omega = \sqrt{\mathcal{M}B_0/J}/\Omega$. One notices that upon increasing the strength of the fix field B_0, the lock-in islands p/q expand in the resonance. Thus, the increase of the frequency ω_l is compensated by the decrease of the oscillation frequency with its amplitude. As the lock-ins are stronger when the rational is smaller, the perturbation that they bring is small when $\omega_l/\Omega << 1$ but becomes strong when ω_l/Ω comes close to 1/3 and 1/2. Le separatrics of the resonance corresponds to a trajectory that is extremely sensitive to all perturbations (this is the case of the pendulum upside down). This is also in the vicinity of these trajectories that the oscillation frequency varies the most rapidly and vanishes. This area is the first region in phase space to become chaotic upon increasing $\mathcal{S}$, as one can see on the figure 21.

11.2 Resonance hierarchy

As said above, the lock-ins associated to small rational p/q are stronger, let us estimate the values of $\mathcal{S}$ where these lock-ins occur. Using the same amplitude of the fix and oscillating field then $\omega_l/\Omega = \mathcal{S}/4$. Thus to reach the lock-in 1/3 one needs to set $\mathcal{S} = 4/3$ (2 for the lock-in $1/2$). These values are well above the threshold value where the primary resonances overlap that is in a domain where chaos affects most of the trajectories and only the heart of the lock-in are not chaotic yet. The lock-in $1/3$ shows the first peculiarity : the three islands emerge at finite distance from the heart of the main resonance, contrary to what we observe for the lock-ins $1/5$ or $1/4$, for instance. Moreover the island structure is different from that of the other lock-ins (see fig. 24). The lock-in $1/2$ is even more special : while the other lock-in leave the heart of the resonance unchanged, the $1/2$ lock-in breaks the resonance in two. The mid-point that is the resonance heart becomes a saddle points joining the two center islands as in the figure 25.

11.3 Resonance break-up by period doubling

The scenario that we have just described repeating itself for the secondary resonance. At their heart, when the two frequencies are commensurate : a new lock-in appears corresponding to a synchronization of the oscillation motions. The same phase oscillation is observed. New series of islands appear in the secondary

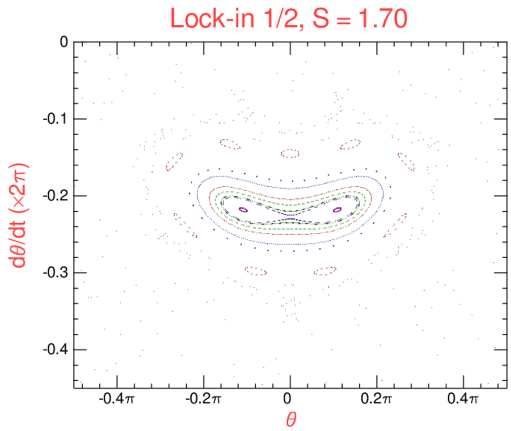

FIGURE 25 – *Compass* POINCARÉ *section at* $\mathcal{S} = 1.70$. *Lock-in* $1/2$, *notice how the lock-in breaks the main resonance in two. Nine small islands now decorate the two secondary resonances, they correspond to a lock-in* $4/9$.

islands. The frequency ratio σ_1 between the phase oscillation and the main frequency increases with the strength of $\mathcal{S}$ until the secondary resonance breaks up leading to a third generation and so on. This cascade of events is in fact a cascade of period doubling.

As we can infer, the phase space resonance structure is hierarchical : one resonance contains secondary resonances that again contain smaller resonances and so forth. The phase space is said to have a self-similar structure.

12 Large scale stochasticity

The scenario that we have sketched is valid at all values of $\mathcal{S}$. A proliferation of resonances is observed, but they do not lead automatically to chaos. Their number but also their strength increase with $\mathcal{S}$. By opening more and more islands, there is less and less available space for regular KAM trajectories. Their shape becomes more and mode perturb and they are finally tearing apart. One way to quantify this fact is to consider the lock-in at the different hierarchical levels. If we consider only passing trajectories, the two primary resonances constitute the first generation. Imagine that we evaluate the trajectory corresponding to the irrational number σ, daughter resonances will appear on each side of this trajectory. One can iterate the process by considering these secondary resonances as a new base and iterate the process. As we go on, the strength of the secondary resonances will either increase or decrease with the iteration order. If they decease, the trajectory will appear smooth at small scale, they are regular. If they increase, the trajectory perturbation grows at each scale leading to a break-up of the trajectory and to a chaotic behavior. This phenomenon corresponds to the breaking of KAM tori.

12.1 Phase space appearance

The local study that we have just sketched allows describing the general shape of the phase space : an intricate structure of lock-ins of regular trajectories near their heart and of chaotic trajectories at their separatrics. The amplification mechanism of resonances and the KAM torus destruction leads to extension of chaotic domains. Let us also notice that regular trajectories are clearly separated in phase space while chaotic trajectories are entangled and form a sea of stochasticity as long as they are not separated by a regular trajectory that constitutes a leakproof boundary.

12.2 Disappearance of the last KAM torus

Thus as long as a KAM torus remains between the two major resonances of the compass, the chaotic trajectories that have soon appeared in close to the separatrics, are not connected and remain bound close to their lock-ins. When the last torus is teared apart, the chaos that emerges is said of *wide scale*. For the compass one can then observe that the chaotic behavior explore the fixed field resonance as well as the rotating one. The compass makes a few oscillations around the fixed field and then starts rotating with the rotating field, jumping from one to the other in an erratic manner.

13 Viscous drag effects

Energy dissipation terms are very difficult to avoid for a real system, what is their effect on the chaotic behaviors that we have described so far ? Viscous drag will lead to the decrease of the amplitude of the oscillation of the pendulum bringing its representative point to the origin of the phase space. Thus dissipative terms will at least restrict the range of initial conditions selecting a small number of them close to the center points corresponding to the minimum of energy. Thus, dissipative terms may be viewed as a simplification compared with Hamiltonian systems for which each initial condition determines a specific trajectory.

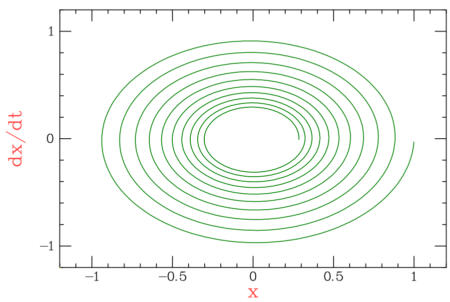

FIGURE 26 – *Damped oscillator trajectories. The area contracting in phase space leads all initial conditions to the origin..*

13.1 The damped oscillator

Let us illustrate the phenomenon for a linear system where we are able to exactly compute the effect : the damped harmonic oscillator.

$$m\ddot{x} + \gamma\dot{x} + kx = 0\,.$$

The solutions take the form :

$$x_0(t) = x_{t=0}\exp\left(\frac{-\gamma t}{2m}\right)\cos\left(\frac{\sqrt{4km-\gamma^2}}{2m}t+\phi\right)$$

When $4km > \gamma^2$, the trajectory spirals towards the origin of the phase space. If $4km < \gamma^2$, the trajectory does not oscillate anymore but converges rapidly towards the origin (overdamped oscillator).

13.2 Area contracting in phase space

In general, for dissipative systems without external energy supply, all points of phase space converge to the origin. All volumetric elements of the phase space are contracting to one point during this evolution.

13.3 Strange attractor and fractal dimension

Deterministic Chaos survives dissipation, once established it signature consists in a special form in phase space called *strange attractor*. An example is given in the figure 27. This corresponds to the POINCARÉ section of a particle in a two well potential subjected to a periodic excitation.

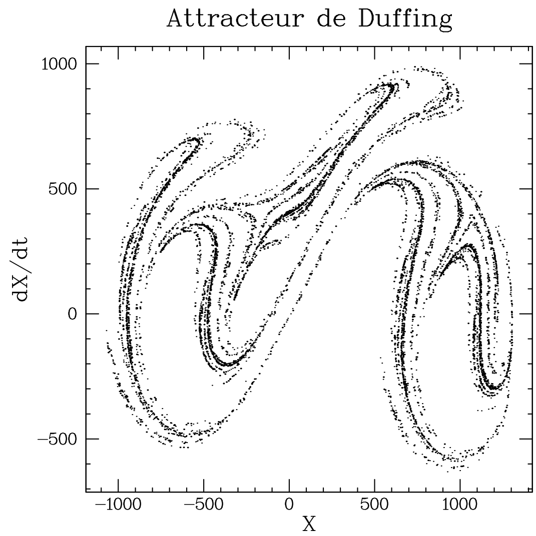

FIGURE 27 – *Strange attractor obtained with an electronic circuit mimicking a particle in a double well potential excited by a sinusoidal perturbation.*

The attractor name comes from the fact that the object attracts the trajectories in phase space. If we impose two different initial conditions, we obtain two trajectories leading to the same figures with the same strange shape but with different point patterns.

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