# DIRECT OBSERVATION OF FIELD QUANTIZATION

**Reference :** M.Brune, F.Schmidt-Kaler, A.Maali, J.Dreyer, E.Hagley, J-M.Raimond and S.Haroche : Phys.Rev.Lett. **76,** 1800 (1996)

We consider here a two-level atom interacting with a single mode of the electromagnetic field. When this mode is treated quantum mechanically, specific features occur in the atomic dynamics, such as damping and revivals of the Rabi oscillations.

## 1. Quantization of a Mode of the Electromagnetic Field

We recall that in *classical mechanic,* a harmonic oscillator of mass $m$ and frequency $\frac{\omega}{2\pi}$ obeys the equation of motion $\begin{cases} \frac{dx}{dt} = \frac{p}{m} \\ \frac{dp}{dt} = -m\omega^2 x \end{cases}$ where $x$ is the position and $p$ the momentum of the oscillator. Defining the *reduced* variables $\begin{cases} X(t) = x(t)\sqrt{\frac{m\omega}{\hbar}} \\ P(t) = \frac{p(t)}{\sqrt{m\hbar\omega}} \end{cases}$, the equations of motion of the oscillator are $\begin{cases} \frac{dX}{dt} = \omega P \\ \frac{dP}{dt} = -\omega X \end{cases}$ (1) and the total energy $U(t)$ is given by :

$$U(t) = \frac{\hbar\omega}{2}\left[X^2(t) + P^2(t)\right] \quad (2)$$

**1.1.** Consider a cavity for electromagnetic waves, of volume $V$. Throughout this problem, we consider a single mode of the electromagnetic field, of the form

$$\vec{E}(\vec{r},t) = \vec{u}_x\, e(t)\sin kz \qquad \vec{B}(\vec{r},t) = \vec{u}_y\, b(t)\cos kz$$

where $\vec{u}_x, \vec{u}_y$ and $\vec{u}_z$ are an orthogonal basis. We recall Maxwell's equations in vacuum :

$$\begin{cases} \nabla.\vec{E}(\vec{r},t) = 0 \qquad \nabla\wedge\vec{E}(\vec{r},t) = -\frac{\partial\vec{B}(\vec{r},t)}{\partial t} \\ \nabla.\vec{B}(\vec{r},t) = 0 \qquad \nabla\wedge\vec{B}(\vec{r},t) = \frac{1}{c^2}\frac{\partial\vec{E}(\vec{r},t)}{\partial t} \end{cases}$$

and the total energy $U(t)$ of the field in the cavity :

$$U(t) = \int_V d^3r\left(\frac{\varepsilon_0}{2}E^2(\vec{r},t) + \frac{1}{2\mu_0}B^2(\vec{r},t)\right) \quad \text{with } \varepsilon_0\mu_0 c^2 = 1 \quad (3)$$

**a)** Express $\frac{de}{dt}$ and $\frac{db}{dt}$ in terms of $k, c, e(t), b(t)$.

**b)** Express $U(t)$ in terms of $V, e(t), b(t), \varepsilon_0, \mu_0$. On can take $\int_V \sin^2 z\; d^3r = \int_V \cos^2 z\; d^3r = \frac{V}{2}$.

**c)** Setting $\omega = kc$ and introducing the reduced variables

$$\chi(t)=\sqrt{\frac{\varepsilon_0 V}{2\hbar\omega}}e(t) \qquad \Pi(t)=\sqrt{\frac{V}{2\mu_0\hbar\omega}}b(t)$$

show that the equations for $\frac{d\chi}{dt},\frac{d\Pi}{dt}$ and $U(t)$ in terms of $\chi,\Pi$ and $\omega$ are formally identical to equations (1) and (2).

**1.2.** The quantization of the mode of the electromagnetic field under consideration is performed in the same way as that of an ordinary harmonic oscillator. One associates to the physical quantities $\chi$ and $\Pi$, Hermitic operators $\hat{\chi}$ and $\hat{\Pi}$ which satisfy the commutation relation

$$\left[\hat{\chi},\hat{\Pi}\right]=i$$

The Hamiltonian of the field in the cavity is

$$\hat{H}_C=\frac{\hbar\omega}{2}\left(\hat{\chi}^2+\hat{\Pi}^2\right)$$

The energy of the field is quantified : $E_n=\left(n+\frac{1}{2}\right)\hbar\omega$ ($n$ is a non-negative integer) ; one denotes by $|n\rangle$ the eigenstate of $\hat{H}_C$ with eigenvalue $E_n$.

The *quantum states* of the field in the cavity are linear combinations of the set $\{|n\rangle\}$. The state $|0\rangle$, of energy $E_0=\frac{\hbar\omega}{2}$ is called the « vacuum », and the state $|n\rangle$ of energy $E_n=E_0+n\hbar\omega$ is called the « $n$ photon state ». A « photon » corresponds to an elementary excitation of the field, of energy $\hbar\omega$. One introduces the « creation » and « annihilation » operators of a photon as

$$\hat{a}^\dagger=\frac{1}{\sqrt{2}}\left(\hat{\chi}-i\hat{\Pi}\right) \text{ and } \hat{a}=\frac{1}{\sqrt{2}}\left(\hat{\chi}+i\hat{\Pi}\right)$$

respectively. These operators satisfy the usual relations :

$$\begin{cases} a^\dagger|n\rangle=\sqrt{n+1}\,|n+1\rangle \\ \hat{a}|n\rangle=\sqrt{n}\,|n-1\rangle \quad \text{if } n\neq 0 \text{ and } \hat{a}|0\rangle=0 \end{cases}$$

**a)** Express $\hat{H}_C$ in terms of $\hat{a}^\dagger$ and $\hat{a}$. The observable $\hat{N}=\hat{a}^\dagger\hat{a}$ is called the « number of photon ».

The *observables* corresponding to the electric and magnetic fields at a point $\vec{r}$ are defined as :

$$\begin{cases} \hat{E}(\vec{r})=\vec{u}_x\sqrt{\frac{\hbar\omega}{\varepsilon_0 V}}\left(\hat{a}+a^\dagger\right)\sin kz \\ \hat{B}(\vec{r})=i\vec{u}_y\sqrt{\frac{\mu_0\hbar\omega}{V}}\left(a^\dagger-\hat{a}\right)\cos kz \end{cases}$$

The interpretation of the theory in terms of states and observables is the same as in ordinary quantum mechanics.

**b)** Calculate the expectation value $\langle\vec{E}(\vec{r})\rangle,\langle\vec{B}(\vec{r})\rangle$ and $\langle n|H_C|n\rangle$ in an $n$-photon state.

**1.3.** The following superposition :

$$|\alpha\rangle=e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle \qquad (4)$$

where $\alpha$ is any complex number, is called « quasi-classical » state of the field.

**a)** Show that $|\alpha\rangle$ is a normalized eigenvector of the annihilation operator $a$ and give the corresponding eigenvalue. Calculate the expectation value $\langle n\rangle$ of the number of photons in that state.

**b)** Show that if, at time $t=0$, the state of the field is $|\Psi(0)\rangle=|\alpha\rangle$, then, at time $t$

$$|\Psi(t)\rangle=e^{-i\frac{\omega t}{2}}\left|\alpha\, e^{-i\omega t}\right\rangle$$

**c)** Calculate the expectation value $\langle\vec{E}(\vec{r})\rangle_t$ and $\langle\vec{B}(\vec{r})\rangle_t$ at time $t$ in a quasi-classical state for which $\alpha$ is real.

**d)** Check that $\langle\vec{E}(\vec{r})\rangle_t$ and $\langle\vec{B}(\vec{r})\rangle_t$ satisfy Maxwell's equations.

**e)** Calculate the energy of a *classical* field such that $\vec{E}_{cl}(\vec{r},t)=\langle\vec{E}(\vec{r})\rangle_t$ and $\vec{B}_{cl}(\vec{r},t)=\langle\vec{B}(\vec{r})\rangle_t$. Compare the result with the expectation value of $\hat{H}_C$ in the same quasi-classical state.

**f)** Why do these results justify the name « quasi-classical » state for $|\alpha\rangle$ if $|\alpha|>>1$ ?

## 2. The Coupling of the Field with an Atom

Consider an atom at point $\vec{r}_0$ in the cavity. The motion of the center of mass of the atom in space is treated classically. Hereafter we restrict ourselves to the *two-dimensional* subspace of internal atomic states generated by the ground state $|f\rangle$ and an excited state $|e\rangle$. The origin of atomic energies is chosen in such a way that the energy of $|f\rangle$ and $|e\rangle$ are respectively $-\frac{\hbar\omega_A}{2}$ and $+\frac{\hbar\omega_A}{2}$ $(\omega_A>0)$. In the basis $\{|f\rangle,|e\rangle\}$, one can introduce the operators :

$$\hat{\sigma}_z=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\qquad \hat{\sigma}_+=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\qquad \hat{\sigma}_-=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$$

that is to say $\hat{\sigma}_+|f\rangle=|e\rangle$ and $\hat{\sigma}_-|e\rangle=|f\rangle$, and the atomic Hamiltonian can be written as :

$$\hat{H}_A=\frac{\hbar\omega_A}{2}\hat{\sigma}_z$$

The set of orthonormal states $\{|f,n\rangle,|e,n\rangle,n\geq 0\}$ where $|f,n\rangle\equiv|f\rangle\otimes|n\rangle$ and $|e,n\rangle\equiv|e\rangle\otimes|n\rangle$ forms a basis of the Hilbert space of the $\{\text{atom}+\text{photons}\}$ states.

**2.1.** Check that it is an eigenbasis of $\hat{H}_0=\hat{H}_A+\hat{H}_C$, and give the corresponding eigenvalues.

**2.2.** In *the remaining parts of the problem* we assume that the frequency of the cavity is exactly tuned to the Bohr frequency of the atom, i.e. $\omega=\omega_A$. Draw schematically the positions of the first 5 energy levels of $\hat{H}_0$. Show that, except for the ground state, the eigenstates of $\hat{H}_0$ are grouped in degenerate pairs.

**2.3.** The Hamiltonian of the electric dipole coupling between the atom and the field can be written as :

$$\hat{W}=\gamma\left(\hat{a}\hat{\sigma}_++\hat{a}^\dagger\hat{\sigma}_-\right)$$

where $\gamma=-d\sqrt{\frac{\hbar\omega}{\varepsilon_0 V}}\sin kz_0$, and where the electric dipole moment $d$ is determined experimentally.

**a)** Write the action of $W$ on the states $|f,n\rangle$ and $|e,n\rangle$.

**b)** To which physical processes do $\hat{a}\hat{\sigma}_+$ and $\hat{a}^\dagger\hat{\sigma}_-$ correspond ?

**2.4.** Determine the eigenstates of $\hat{H} = \hat{H}_0 + \hat{W}$ and the corresponding energies. Show that the problem reduces the diagonalisation of a set of 2x2 matrices. One hereafter sets :

$$\left|\Phi_n^\pm\right\rangle = \frac{1}{\sqrt{2}}\left(\left|f, n+1\right\rangle \pm \left|e, n\right\rangle\right)$$

$$\frac{\hbar\Omega_0}{2} = \gamma = -d\sqrt{\frac{\hbar\omega}{\varepsilon_0 V}}\sin kz_0 \qquad \Omega_n = \Omega_0\sqrt{n+1}$$

The energies corresponding to eigenstates $\left|\Phi_n^\pm\right\rangle$ are denoted $E_n^\pm$..

## 3. Interaction of the Atom and an « Empty » Cavity

*In the following one assumes that the atom crosses the cavity along a line where* $\sin kz_0 = 1$.

An atom in the excited state $|e\rangle$ is sent into the cavity prepared in the vacuum state $|0\rangle$. At time $t = 0$, when the atom enters the cavity, the state of the system is $|e, n = 0\rangle$.

**3.1.** What is the state of the system at a later time $t$ ?

**3.2.** What is the probability $P_f(T)$ of finding the atom in the state $f$ at time $T$ when the atom leaves the cavity ? Show that $P_f(T)$ is a periodic function of $T$ ($T$ is varied by changing the velocity of the atom).

**3.3.** The experiment has been performed on rubidium atoms for a couple of states $(f, e)$ such that $d = 1{,}1\times10^{-26}\,\mathrm{C.m}$ and $\frac{\omega}{2\pi} = 5{,}0\times10^{10}\,\mathrm{Hz}$. The volume of the cavity is $V = 1{,}87\times10^{-6}\,\mathrm{m}^3$ (we recall that $\varepsilon_0 = \frac{1}{36\pi\times10^9}$SI). The curve $P_f(T)$, together with the real part of its Fourier transform $J(\nu) = \int_0^\infty \cos(2\pi\nu T)P_f(T)dT$, are shown in Fig.1. One observes a damped oscillation, the damping being due to imperfections in the experimental setup.

How do theory and experiment compare ?

(We recall that the Fourier transform of a damped sinusoid in time exhibits a peak at the frequency of this sinusoid, whose width is proportional to the inverse of the characteristic damping time.)

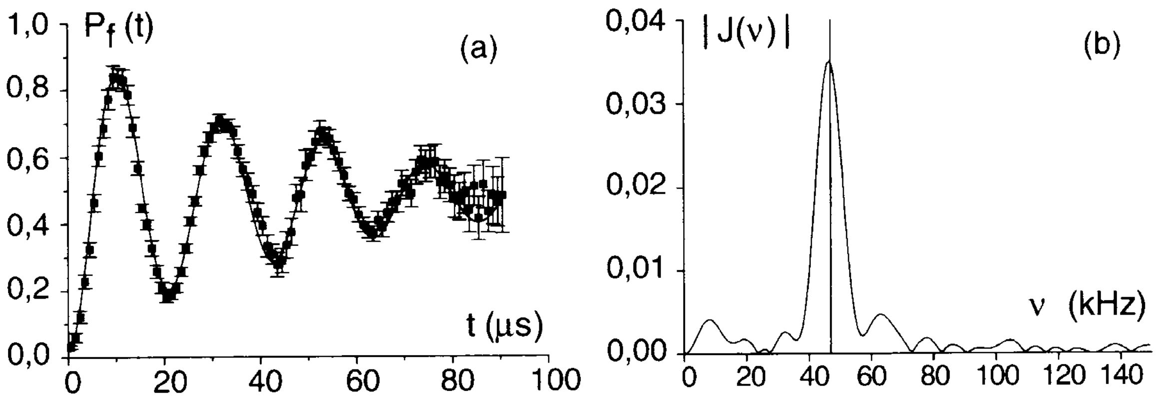


**Fig.1 (a)** Probability $P_f(T)$ of detecting the atom in the ground state after it crosses a cavity containing zero photons ; **(b)** Fourier transform of this probability, as defined in the text.

## 4. Interaction of an Atom with a Quasi-Classical State

The atom, initially in the state $|e\rangle$, is now sent into a cavity where a quasi-classical state $|\alpha\rangle$ of the field has been prepared. At time $t = 0$ the atom enters the cavity and the state of the system is $|e\rangle \otimes |\alpha\rangle$.

**4.1.** Calculate the probability $P_f(T, n)$ of finding, at time $T$, the atom in the state $|f\rangle$ and the field in the state $|n+1\rangle$, for $n \geq 0$. What is the probability of finding the atom in the state $|f\rangle$ and the field in the state $|0\rangle$ ?

**4.2.** Write the probability $P_f(T)$ of finding the atom in the state $|f\rangle$, independently of the state of the field, as an infinite sum of oscillating functions.

**4.3.** On Fig.2 are plotted an experimental measurement of $P_f(T)$ and the real part of its Fourier transform $J(\nu)$. The cavity used for this measurement is the same as Fig.1, but the field has been prepared in a quasi-classical state before the atom is sent in.

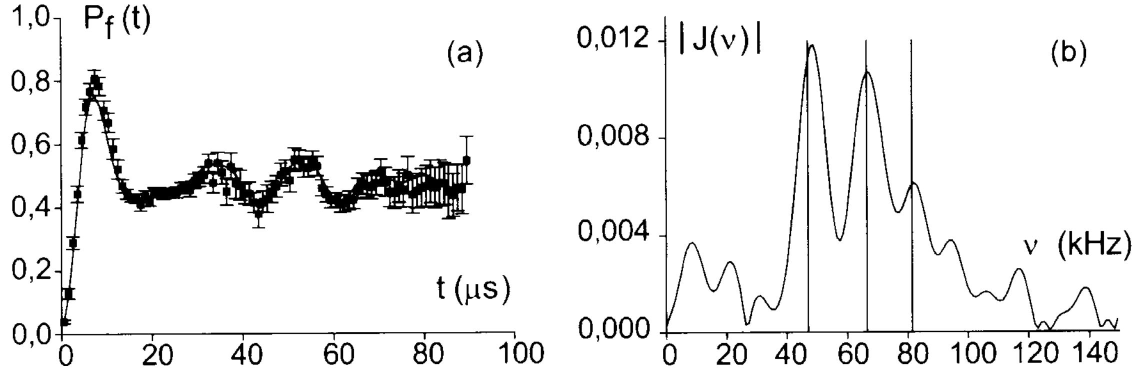


**Fig.2 a)** Probability $P_f(T)$ of measuring the atom in the ground state after the atom is passed through a cavity containing a quasi-classical State of the electromagnetic field ; **b)** Fourier transform of this probability.

**a)** Determine the three frequencies $\nu_0, \nu_1, \nu_3$ which contribute most strongly to $P_f(T)$.

**b)** Do the ratio $\frac{\nu_1}{\nu_0}$ and $\frac{\nu_2}{\nu_0}$ have the expected values ?

**c)** From the values $J(\nu_0)$ and $J(\nu_1)$, determine an approximate value for the mean number of photon $|\alpha|^2$ in the cavity.

## 5. Large Numbers of Photons : Damping and Revivals

Consider a quasi-classical state $|\alpha\rangle$ of the field corresponding to a large mean number of photons : $|\alpha|^2 \approx n_0 >> 1$, where $n_0$ is an integer. In this case, the probability $\pi(n)$ of finding $n$ photons can be cast, in good approximation, in the form :

$$\pi(n) = e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!} \approx \frac{1}{\sqrt{2\pi n_0}} \exp\left[ -\frac{(n-n_0)^2}{2n_0} \right]$$

This gaussian limit of the Poisson distribution can be observed by using the Stirling formula $n! \approx n^n e^{-n} \sqrt{2\pi n}$ and expanding $\ln \pi(n)$ in the vicinity of $n = n_0$.

**5.1.** Show that this probability takes significant values only if $n$ is in a neighborhood $\delta n$ of $n_0$. Give the relative value $\dfrac{\delta n}{n_0}$.

**5.2.** For such a quasi-classical state, one tries to evaluate the probability $P_f(T)$ of detecting the atom in the state $f$ after its interaction with the field. In order to do this,

- one linearizes the dependence of $\Omega_n$ on $n$ in the vicinity of $n_0$ :

$$\Omega_n \approx \Omega_{n_0} + \Omega_0 \frac{n-n_0}{2\sqrt{n_0+1}} \qquad (5)$$

- one replaces the discrete summation in $P_f(T)$ by an integral.

**a)** Show that, under these approximations, $P_f(T)$ is an oscillating function of $T$ for short times, but that this oscillations is damped away after a characteristic time $T_D$. Give the value of $T_D$.

We recall that : $\displaystyle\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-x_0)^2}{2\sigma^2}} \cos(\alpha x)dx = e^{-\frac{\alpha^2\sigma^2}{2}} \cos(\alpha x_0)$

**b)** Does this damping time depend on the mean value of the number of photon $n_0$ ?

**c)** Give a qualitative explanation for this damping.

**5.3.** If one keeps the expression of $P_f(T)$ as a discrete sum, an exact numerical calculation shows that one expects a revival of the oscillations of $P_f(T)$ for certain times $T_R$ large compared to $T_D$, as shown in Fig.3. This phenomenon is called *quantum revival* and it currently studied experimentally.

Keeping the discrete sum but using the approximation (5), can you explain the revival qualitatively ? How does the time of the first revival depend on $n_0$ ?

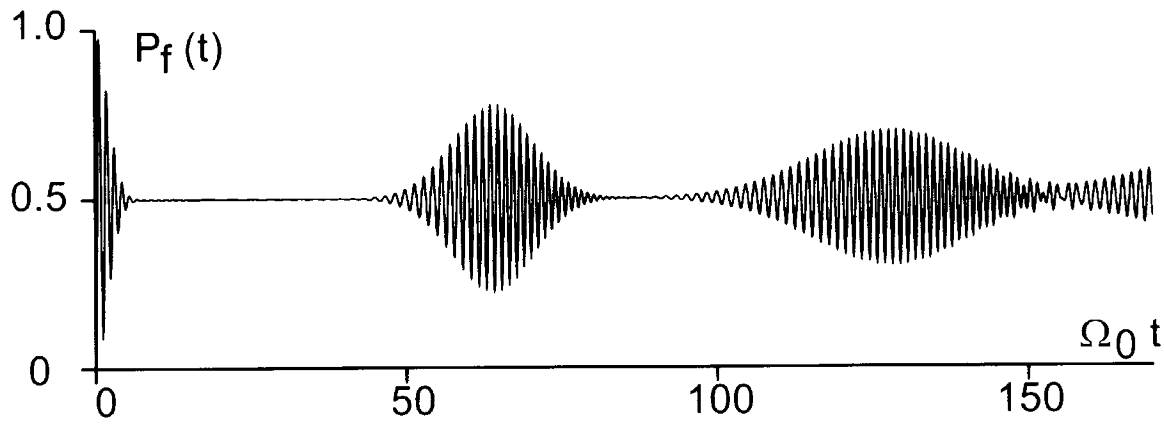


**Fig.3.** Exact theoretical calculation of $P_f(T)$ for $\langle n \rangle \approx 25$ photons.

# Solutions

## 1.

**1.1. a)** The pair of Maxwell equations $\nabla.\vec{E}=0$ and $\nabla.\vec{B}=0$ are satisfied whatever the values of the functions $e(t)$ and $b(t)$. The equations $\nabla\wedge\vec{E}=-\dfrac{\partial\vec{B}}{\partial t}$ and $c^2\nabla\wedge\vec{B}=-\dfrac{\partial\vec{E}}{\partial t}$ require that :

$$\frac{de}{dt}=c^2kb(t) \qquad \frac{db}{dt}=-ke(t)$$

**b)** The electromagnetic energy can be written as :

$$U(t)=\int_V\left(\frac{\varepsilon_0}{2}e^2(t)\sin^2 kz+\frac{1}{2\mu_0}b^2(t)\cos^2 kz\right)d^3r=\frac{\varepsilon_0 V}{4}e^2(t)+\frac{V}{2\mu_0}b^2(t)$$

**c)** Under the change of functions suggested in the text, we obtain :

$$\begin{cases}\dot{\chi}=\omega\Pi\\ \dot{\Pi}=-\omega\chi\end{cases} \qquad U(t)=\frac{\hbar\omega}{2}\left(\chi^2(t)+\Pi^2(t)\right)$$

These two equations are formally identical to the equations of motion of a particle in a harmonic oscillator potential.

**1.2. a)** From $\left[\hat{\chi},\hat{\Pi}\right]=i$ we deduce that :

$$\left[\hat{a},\hat{a}^\dagger\right]=\frac{1}{2}\left[\hat{\chi}+i\hat{\Pi},\hat{\chi}-i\hat{\Pi}\right]=1$$

In addition, $\hat{\chi}=\dfrac{\hat{a}+\hat{a}^\dagger}{\sqrt{2}}$ and $\hat{\Pi}=i\dfrac{\hat{a}^\dagger-\hat{a}}{\sqrt{2}}$ i.e. : $\hat{H}_C=\dfrac{\hbar\omega}{2}\left(\hat{a}\hat{a}^\dagger+\hat{a}^\dagger\hat{a}\right)=\hbar\omega\left(\hat{a}^\dagger\hat{a}+\dfrac{1}{2}\right)$ or

$$\hat{H}_C=\hbar\omega\left(\hat{N}+\frac{1}{2}\right)$$

**b)** For an $n$ photon state, we find $\langle n|\hat{a}|n\rangle=\langle n|\hat{a}^\dagger|n\rangle=0$, which results in

$$\left\langle\vec{E}(\vec{r})\right\rangle=\left\langle\vec{B}(\vec{r})\right\rangle=0$$

The state $|n\rangle$ is an eigenstate of $H_C$ with eigenvalue $\left(n+\dfrac{1}{2}\right)\hbar\omega$, i.e.

$$\left\langle\hat{H}_C\right\rangle=\left(n+\frac{1}{2}\right)\hbar\omega$$

**1.3. a)** The action of $\hat{a}$ on $|\alpha\rangle$ gives

$$\begin{aligned}\hat{a}|\alpha\rangle&=e^{-\frac{|\alpha|^2}{2}}\sum_{n=1}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\sqrt{n}|n-1\rangle\\ &=\alpha\, e^{-\frac{|\alpha|^2}{2}}\sum_{n=1}^{\infty}\frac{\alpha^{n-1}}{\sqrt{(n-1)!}}|n-1\rangle=\alpha|\alpha\rangle\end{aligned}$$

The vector $|\alpha\rangle$ is normalized

$$\langle\alpha|\alpha\rangle = e^{-|\alpha|^2}\sum_{n=0}^{\infty}\frac{\left(\alpha^*\right)^n\alpha^n}{n!} = 1$$

The expectation value of the number of photons in that state is :

$$\langle n\rangle = \langle\alpha|\hat{N}|\alpha\rangle = \langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle = \left\|\hat{a}|\alpha\rangle\right\|^2 = |\alpha|^2$$

**b)** The time evolution of $|\Psi(t)\rangle$ is given by

$$|\Psi(t)\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}e^{-i\omega\left(n+\frac{1}{2}\right)t}|n\rangle$$

$$= e^{-\frac{i\omega t}{2}}e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\left(\alpha e^{-i\omega t}\right)^n}{\sqrt{n!}}|n\rangle$$

$$= e^{-\frac{i\omega t}{2}}\left|\left(\alpha e^{-i\omega t}\right)\right\rangle$$

**c)** The expectation values of the electric and magnetic fields are

$$\left\langle\vec{E}(\vec{r})\right\rangle_t = 2\alpha\cos\omega t\sin kz\sqrt{\frac{\hbar\omega}{\varepsilon_0 V}}\,\vec{u}_x$$

$$\left\langle\vec{B}(\vec{r})\right\rangle_t = -2\alpha\sin\omega t\cos kz\sqrt{\frac{\hbar\omega\mu_0}{V}}\,\vec{u}_y$$

**d)** These fields are the same type as the classical fields considered of the beginning of the problem, with

$$e(t) = 2\alpha\sqrt{\frac{\hbar\omega}{\varepsilon_0 V}}\cos\omega t \qquad b(t) - 2\alpha\sqrt{\frac{\hbar\omega\mu_0}{V}}\sin\omega t$$

Given the relation $\varepsilon_0\mu_0 c^2 = 1$, we verify that $\dot{e}(t) = c^2 k b(t)$ and $\dot{b}(t) = -ke(t)$. Therefore the expectation values of the field operators satisfy Maxwell's equations.

**e)** The energy of the classical field can be calculated using the results of question **1.** Since $\cos^2\omega t + \sin^2\omega t = 1$, we find $U(t) = \hbar\omega\alpha^2$. This "classical" energy is therefore time-independent. The expectation value of $H_C$ is :

$$\left\langle\hat{H}_C\right\rangle = \left\langle\hbar\omega\left(\hat{N}+\frac{1}{2}\right)\right\rangle = \hbar\omega\left(\alpha^2+\frac{1}{2}\right)$$

It is also time-independent (Ehrenfest's theorem)

**f)** For $|\alpha|$ much larger than 1, the ratio $\dfrac{U(t)}{\left\langle\hat{H}_C\right\rangle}$ is close to 1. More generally, the expectation value of a physical quantity as calculated for a quantum field in the state $|\alpha\rangle$, will be close to the value calculated for a classical field such that $\vec{E}_{Cl}(\vec{r},t) = \left\langle\vec{E}(\vec{r})\right\rangle_t$ and $\vec{B}_{Cl}(\vec{r},t) = \left\langle\vec{B}(\vec{r})\right\rangle_t$.

**2.**

**2.1.** One checks that

$$\hat{H}_0|f,n\rangle = \left(-\frac{\hbar\omega_A}{2}+\left(n+\frac{1}{2}\right)\hbar\omega\right)|f,n\rangle$$

$$\hat{H}_0|e,n\rangle = \left(\frac{\hbar\omega_A}{2}+\left(n+\frac{1}{2}\right)\hbar\omega\right)|e,n\rangle$$

For a cavity which resonates at the atom's frequency, i.e. if $\omega = \omega_A$, the couple of states $|f, n+1\rangle, |e, n\rangle$ are degenerate. The first five levels of $\hat{H}_0$ are shown Fig. 4.a. Only the ground state $|f, 0\rangle$ of the atom+field is non-degenerate.

**2.3.**

**a)** The action of $\hat{W}$ on the basis of $\hat{H}_0$ is given by

$$\hat{W}|f,n\rangle = \begin{cases} \sqrt{n}\gamma|e,n-1\rangle \text{ if } n \geq 1 \\ 0 \text{ if } n = 0 \end{cases}$$

$$\hat{W}|e,n\rangle = \sqrt{n+1}\gamma|f,n+1\rangle$$

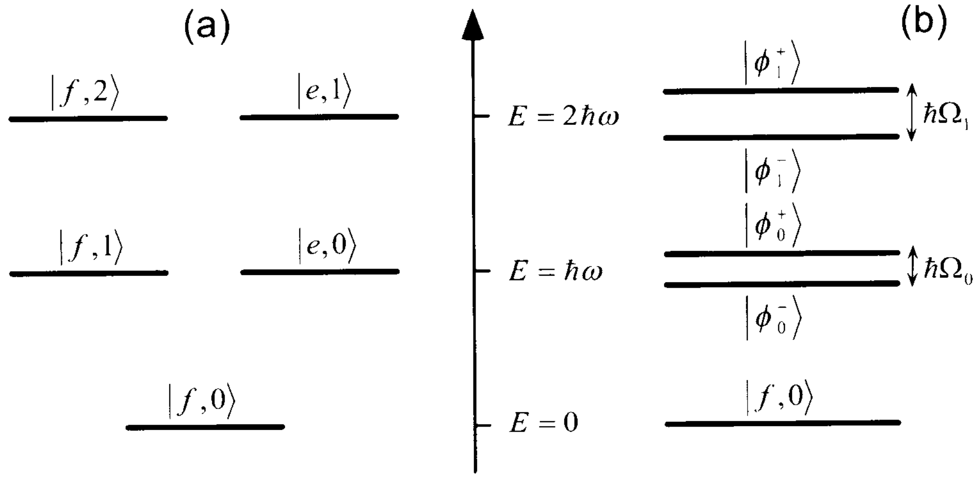


**Fig. 4 (a)** *Positions of the five first energy levels of* $\hat{H}_0$. **(b)** *Positions of the five first energy levels of* $\hat{H} = \hat{H}_0 + \hat{W}$.

The coupling under consideration corresponds to an electric dipole interaction of the form $-\hat{\vec{D}}.\vec{E}(\vec{r})$, where $\hat{\vec{D}}$ is the observable electric dipole moment of the atom.

**b)** $\hat{W}$ couples the two states of each degenerate pair. The term $\hat{a}\hat{\sigma}_+$ correspond to the absorption of a photon by the atom, which undergoes a transition from the ground state to the excited state. The term $a^\dagger\hat{\sigma}_-$ correspond to the emission of a photon by the atom, which undergoes a transition from the excited state to the ground state.

**2.4.** The operator $\hat{W}$ is block-diagonal in the eigenbasis of $\hat{H}_0$ $\{|f,n\rangle, |e,n\rangle\}$, therefore :

- The state $|f,0\rangle$ is an eigenstate of $\hat{H} = \hat{H}_0 + \hat{W}$ with the eigenvalue 0.
- In each eigen-subspace of $\hat{H}_0$ generated by $\{|f,n+1\rangle, |e,n\rangle\}$ with $n \geq 0$ one must diagonalize the 2x2 matrix

$$\begin{pmatrix} (n+1)\hbar\omega & \frac{\hbar\Omega_n}{2} \\ \frac{\hbar\Omega_n}{2} & (n+1)\hbar\omega \end{pmatrix}$$

whose eigenvectors and corresponding eigenvalues are $(n \geq 0)$

$$\left|\Phi_n^+\right\rangle \text{ corresponding to } E_n^+ = (n+1)\hbar\omega + \frac{\hbar\Omega_n}{2}$$

$$\left|\Phi_n^-\right\rangle \text{ corresponding to } E_n^- = (n+1)\hbar\omega - \frac{\hbar\Omega_n}{2}$$

The first energy levels of $\hat{H} = \hat{H}_0 + \hat{W}$ are shown in Fig.4.b.

**3.**

**3.1.** We expand the initial state on the eigenbasis of $\hat{H}$

$$\left|\Psi_0\right\rangle = \left|e,0\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\Phi_0^+\right\rangle - \left|\Phi_0^-\right\rangle\right)$$

The time evolution of the state vector is therefore given by

$$\left|\Psi(t)\right\rangle = \frac{1}{\sqrt{2}}\left(e^{-\frac{i}{\hbar}E_0^+ t}\left|\Phi_0^+\right\rangle - e^{-\frac{i}{\hbar}E_0^- t}\left|\Phi_0^-\right\rangle\right)$$

$$= \frac{e^{-i\omega t}}{\sqrt{2}}\left(e^{-i\frac{\Omega_0 t}{2}}\left|\Phi_0^+\right\rangle - e^{i\frac{\Omega_0 t}{2}}\left|\Phi_0^-\right\rangle\right)$$

**3.2.** In general, the probability of detecting the atom in the state $f$, independently of the field state is given by :

$$P_f(T) = \sum_{n=0}^{\infty}\left|\left\langle f,n \middle| \Psi(T)\right\rangle\right|^2$$

In the particular case if an initially empty cavity, only the term $n=1$ contributes to the sum. Using $\left|f,1\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\Phi_0^+\right\rangle + \left|\Phi_0^-\right\rangle\right)$ we find

$$P_f(T) = \sin^2\frac{\Omega_0 T}{2} = \frac{1}{2}\left(1 - \cos\Omega_0 T\right)$$

It is indeed a periodic function of $T$, with angular frequency $\Omega_0$.

**3.3.** Experimentally, one measures an oscillation of frequency $\nu_0 = 47\text{kHz}$

This result correspond to the expected value $\nu_0 = \frac{1}{2\pi}\frac{2d}{\hbar}\sqrt{\frac{\hbar\omega}{\varepsilon_0 V}}$

**4.**

**4.1.** Again, we expand the initial state on the eigenbasis of $H_0 + W$

$$\left|\Psi(0)\right\rangle = \left|e\right\rangle \otimes \left|\alpha\right\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\left|e,n\right\rangle$$

$$= e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\frac{1}{\sqrt{2}}\left(\left|\Phi_n^+\right\rangle - \left|\Phi_n^-\right\rangle\right)$$

At time $t$ the state vector is

$$\left|\Psi(t)\right\rangle == e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\frac{1}{\sqrt{2}}\left(e^{-\frac{i}{\hbar}E_n^+ t}\left|\Phi_n^+\right\rangle - e^{-\frac{i}{\hbar}E_n^- t}\left|\Phi_n^-\right\rangle\right)$$

we therefore observe that :

- the probability of finding the atom in the state $\left|f\right\rangle$ and the field in the state $\left|0\right\rangle$ vanishes for all value of $T$

• the probability $P_f(T,n)$ can be obtained from the scalar product of

$\left|\Psi(T)\right\rangle$ and $\left|f,n+1\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\Phi_n^+\right\rangle + \left|\Phi_n^-\right\rangle\right)$

$$P_f(T,n) = \frac{1}{4}e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!}\left|e^{-\frac{i}{\hbar}E_n^+T} - e^{-\frac{i}{\hbar}E_n^-T}\right|^2$$

$$= e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!}\sin^2\frac{\Omega_n T}{2} = \frac{1}{2}e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!}\left(1-\cos\Omega_n T\right)$$

**4.2.** The probability $P_f(T)$ is simply the sum of all probabilities $P_f(T,n)$ :

$$P_f(T) = \sum_{n=0}^{\infty} P_f(T,n) = \frac{1}{2} - \frac{e^{-|\alpha|^2}}{2}\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!}\cos\Omega_n T$$

**4.3.**

**a)** The three most prominent peaks of $J(\nu)$ occur at the frequencies $\nu_0 = 47\text{kHz}$ (already found for an empty cavity), $\nu_1 = 65\text{kHz}$ and $\nu_2 = 81\text{kHz}$.

**b)** The ratios of the measured frequencies are very close the theoretical predictions $\frac{\nu_1}{\nu_0} = \sqrt{2}$ and $\frac{\nu_2}{\nu_0} = \sqrt{3}$ .

**c)** The ratio $\frac{J(\nu_1)}{J(\nu_0)}$ is of the order of 0,9. Assuming the peaks have the same widths, and that these widths are small compared to the splitting $\nu_1 - \nu_0$, this ratio corresponds to the average number of photons $|\alpha|^2$ in the cavity.

**5.**

**5.1.** The probability $\pi(n)$ takes significant values only if $\frac{(n-n_0)}{2n_0}$ is not much larger than 1,i.e. for integer values of $n$ in a neighborhood of $n_0$ of relative extension of the order of $\frac{1}{\sqrt{n_0}}$. For $n_0 \gg 1$, the distribution $\pi(n)$ is therefore peaked around $n_0$.

**5.2.**

**a)** Consider the result of question **4.2.** where we replace $\Omega_n$ by its approximation (5) :

$$P_f(T) = \frac{1}{2} - \frac{1}{2}\sum_{n=0}^{\infty}\pi(n)\cos\left[\left(\Omega_{n_0} + \Omega_0\frac{n-n_0}{2\sqrt{n_0+1}}\right)T\right]$$

We now replace the discrete sum by an integral :

$$P_f(T) = \frac{1}{2} - \frac{1}{2}\int_{-\infty}^{+\infty}\frac{e^{-\frac{u^2}{2n_0}}}{\sqrt{2\pi n_0}}\cos\left[\left(\Omega_{n_0} + \Omega_0\frac{u}{2\sqrt{n_0+1}}\right)T\right]du$$

we have extended the lower integration bound from $-n_0$ down to $-\infty$, using the fact that the width of the Gaussian is $\sqrt{n_0} \ll n_0$. We now develop the expression to be integrated upon :

$$\cos\left[\left(\Omega_{n_0}+\Omega_0\frac{u}{2\sqrt{n_0+1}}\right)T\right]=\cos\left(\Omega_{n_0}T\right)\cos\left(\frac{\Omega_0 uT}{2\sqrt{n_0+1}}\right)-\sin\left(\Omega_{n_0}T\right)\sin\left(\frac{\Omega_0 uT}{2\sqrt{n_0+1}}\right)$$

The sine term does not contribute to the integral (odd function) and we find

$$P_f\left(T\right)=\frac{1}{2}-\frac{1}{2}\cos\left(\Omega_{n_0}T\right)\exp\left(-\frac{\Omega_0^2T^2n_0}{8\left(n_0+1\right)}\right)$$

For $n_0 >> 1$, the argument of the exponential simplifies, and we obtain :

$$P_f\left(T\right)=\frac{1}{2}-\frac{1}{2}\cos\left(\Omega_{n_0}T\right)\exp\left(-\frac{T^2}{T_D^2}\right)$$

with $T_D=\dfrac{2\sqrt{2}}{\Omega_0}$.

**b)** In this approximation, the oscillations are damped out in a time $T_D$ which is independent of the number of photons $n_0$. For a given atomic transition (for fixed values of $d$ and $\omega$), this time $T_D$ increases like the square root of the volume of the cavity. In the limit of an infinite cavity, i.e. an atom in empty space, the damping time becomes infinite : we recover the usual Rabi oscillation. For a cavity of finite size, the number of visible oscillations of $P_f\left(T\right)$ is roughly $\nu_{n_0}T_D\sim\sqrt{n_0}$.

**c)** The function $P_f\left(T\right)$ is made up of a large number of oscillating functions with similar frequencies. Initially, these different functions are in phase, and their sum $P_f\left(T\right)$ exhibits marked oscillations. After a time $T_D$, the various oscillations are no longer in phase with another and the resulting oscillations of $P_f\left(T\right)$ is damped. One can find the damping time by simply estimating the time which the two frequencies at half width on either side of the maximum of $\pi\left(n\right)$ are out of phase by $\pi$:

$$\Omega_{n_0+\sqrt{n_0}}T_D\sim\Omega_{n_0-\sqrt{n_0}}T_D+\pi \text{ and } \sqrt{n_0\pm\sqrt{n_0}}\simeq\sqrt{n_0}\pm\frac{1}{2}\Rightarrow\Omega_0T_D\sim\pi$$

**5.3.** Within the approximation (5) suggested in the text, equation (6) above corresponds to a periodic evolution of period

$$T_R=\frac{4\pi}{\Omega_0}\sqrt{n_0+1}$$

indeed

$$\left(\Omega_{n_0}+\Omega_0\frac{n-n_0}{2\sqrt{n_0+1}}\right)T_R=4\pi\left(n_0+1\right)+2\pi\left(n-n_0\right)$$

We therefore expect that all the oscillating functions which contribute to $P_f\left(T\right)$ will reset in phase at times $T_R, 2T_R, \cdots$ The time of the first revival, measured in Fig.3 is $\Omega_0T\simeq 64$, in excellent agreement with the prediction. Notice that $T_R\sim 4\sqrt{n_0}T_D$, which means that the revival time is always large compared to the damping time.

Actually, one can see from the result of Fig.3. that the functions are only partly in phase. This comes from the fact that the numerical calculation has been done with the exact expression of $\Omega_n$. In this case, the difference between two consecutive frequencies $\Omega_{n+1}-\Omega_n$ is not exactly a constant, contrary to what happens in approximation (5) ; the function $P_f\left(T\right)$ is not really periodic. After a few revivals, one obtains a complicated behavior of $P_f\left(T\right)$, which can be analyzed with the techniques developed for the study of chaos.

• *The experiment described in this problem have been performed in Paris at the Laboratoire Kastler-Brossel (E.N.S.). The pair of levels* $\left(f,e\right)$ *correspond to very excited levels of rubidium, which explains the large value of the electric dipole moment* $d$ *. He field is confined in a superconducting niobium cavity (*$Q-$factor of $\sim 10^8$*), cooled down to 0,8K in order to avoid perturbations to the experiment due to the thermal blackbody radiation.*